Near-extremal points of a random walk and variations around Odlyzko 's algorithm for the search of its maximum

Grégory Schehr
Labo. de Physique Théorique et Modèles Statistiques (LPTMS)
CNRS-Université Paris-Sud

Séminaire Philippe Flajolet, December 4, 2014

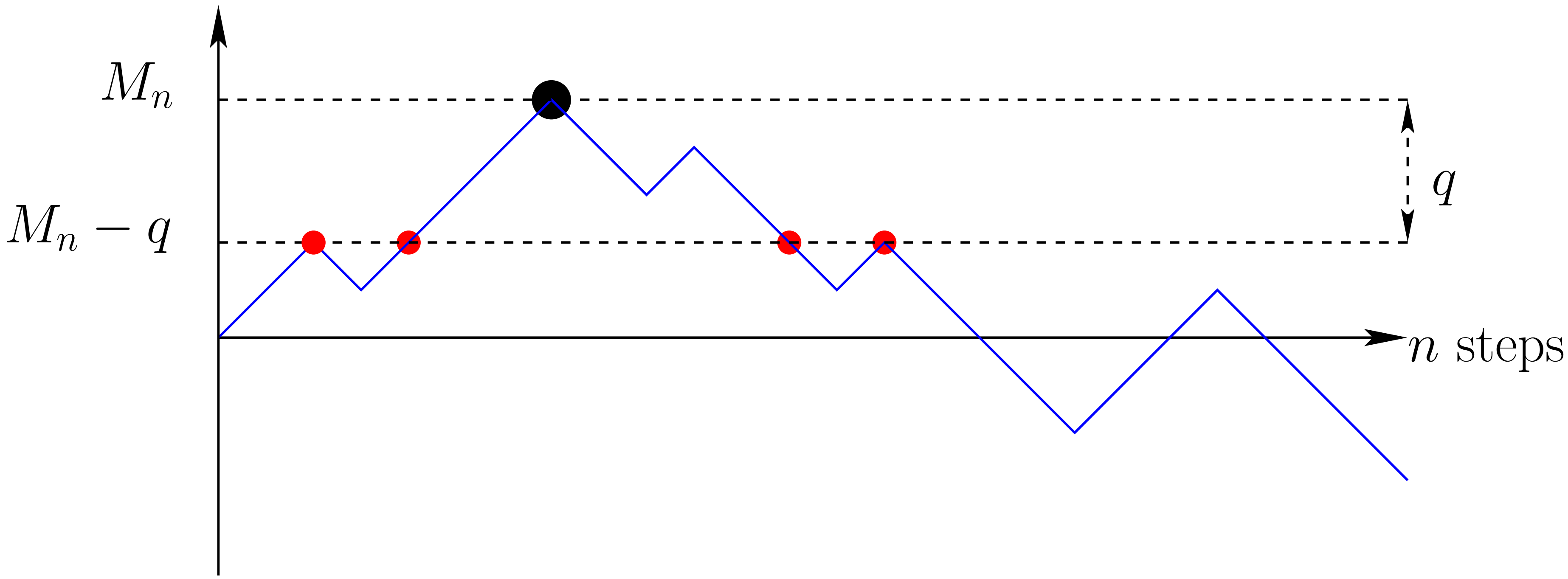
- Anthony Perret (LPTMS, Orsay)
- Alain Comtet (LPTMS, Orsay)
- Satya N. Majumdar (LPTMS, Orsay)

A. Perret, A. Comtet, S. N. Majumdar, G. S., Phys. Rev. Lett. **111**, 240601 (2013)
& preprint arXiv:1502.01218

Near-extremal points of random walks

$$X_0 = 0 \; ,$$

$$X_i = X_{i-1} + \epsilon_i \; , \; \epsilon_i = \begin{cases} + & 1 \; , \; \text{w. proba. } 1/2 \\ - & 1 \; , \; \text{w. proba. } 1/2 \end{cases}$$

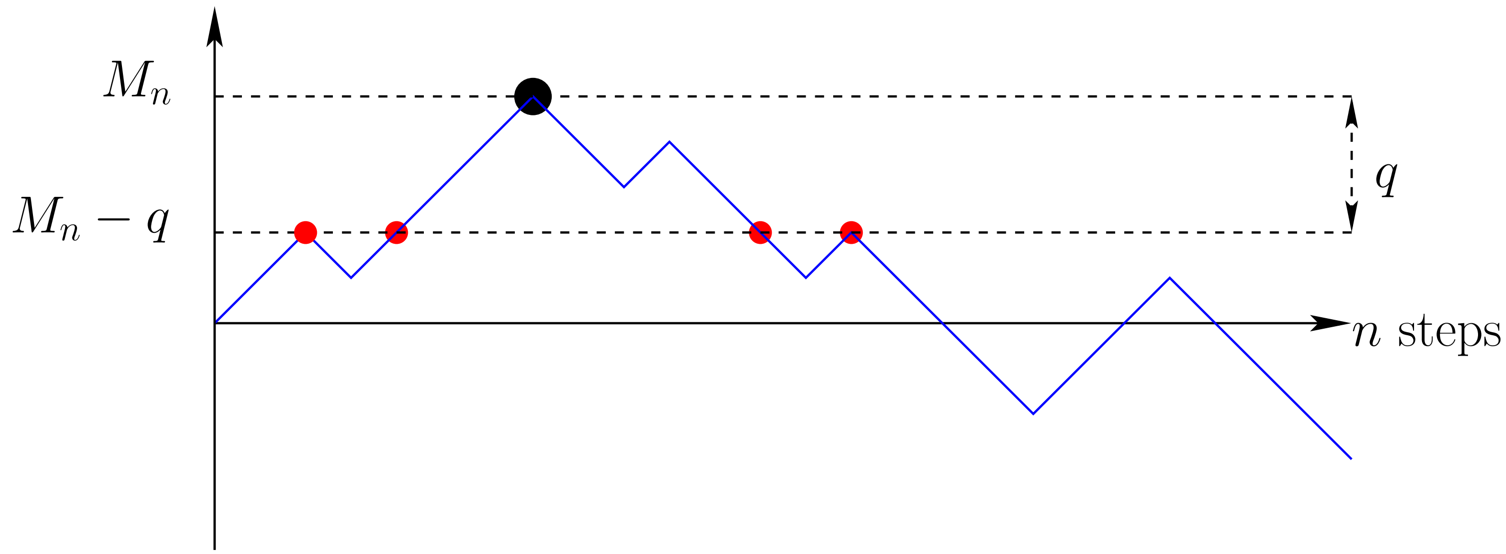


- Number of near-extremal points

$$M_n = \max_{0 \leq k \leq n} X_k$$

$$\boxed{D(q,n) = \# \left\{ 0 \leq k \leq n \,|\, X_k = M_n - q \right\}}$$

Related questions in the literature on random walks



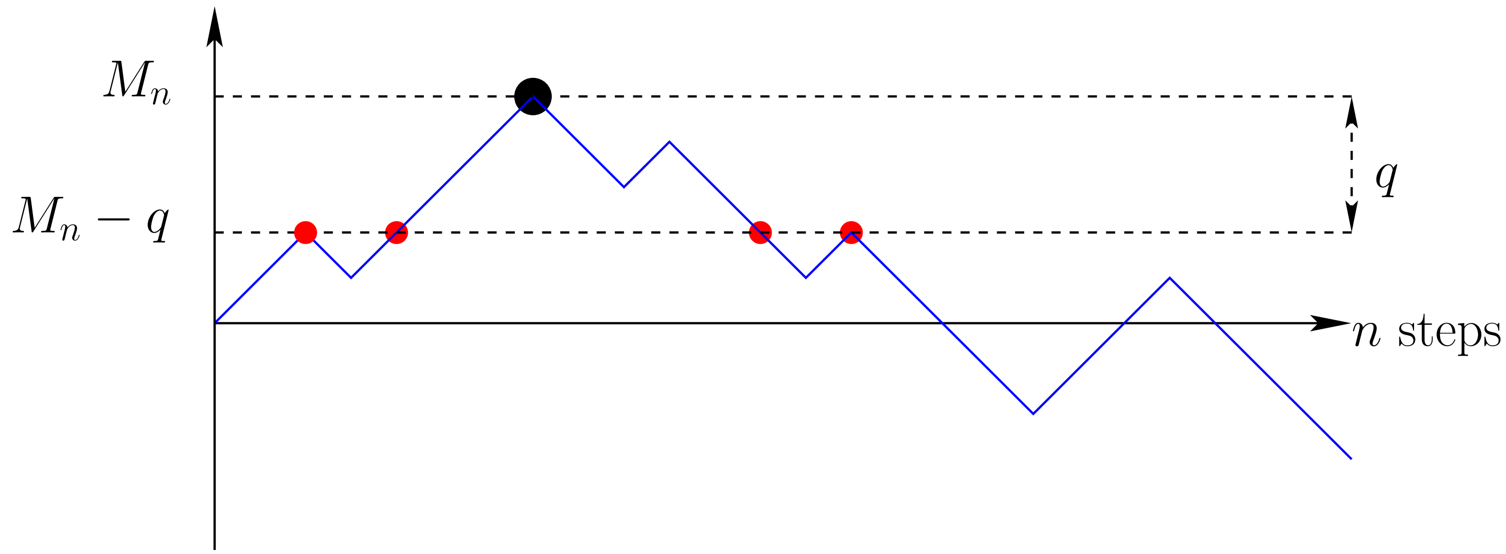
$$D(q,n) = \#\{0 \leq k \leq n \,|\, X_k = M_n - q\}$$

- Local time of RW (and Brownian motion)

- Frequently and rarely visited sites : Erdös, Revesz,..., Toth

- Number of times a random walk is at its maximum: Csaki, Odlyzko

Motivations



$$D(q,n) = \#\{0 \le k \le n \,|\, X_k = M_n - q\}$$

- "Crowding" near the maximum: is the maximum lonely at the top ?

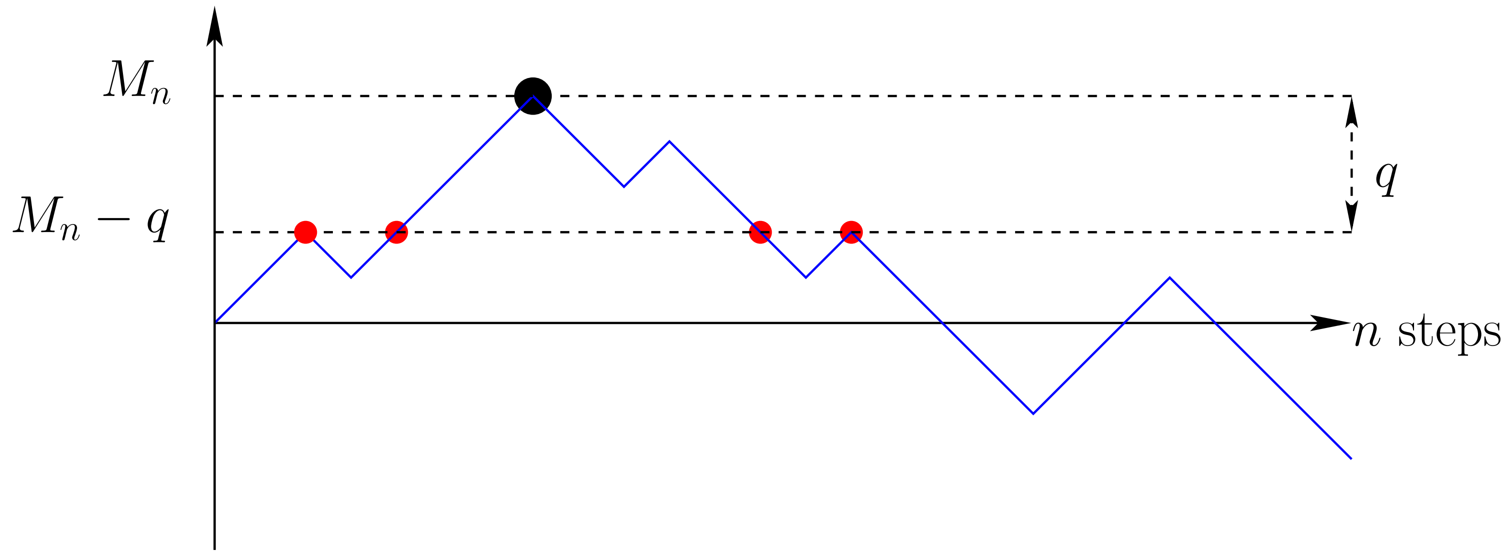
  Sabhapandit, Majumdar '07

- Plays an important role in the analysis of the optimal algorithm to find the maximum of a random walk

  Odlyzko '95
  Hwang '97, Chassaing '99
  Chassaing, Marckert, Yor '99

- Functionals of the maximum of RW and Brownian motion

Local time of Brownian motion close to its maximum



$$D(q,n) = \#\left\{0 \le k \le n \,|\, X_k = M_n - q\right\}$$

- Asymptotic limit $n \to \infty$

$$\frac{1}{\sqrt{n}} D(q = \lfloor r\sqrt{n} \rfloor, n) \xrightarrow[\text{law}]{} \rho(r) = \int_0^1 \delta(x_{\max} - x(\tau) - r)\, d\tau$$

$x(\tau)$: Brownian motion

$$x_{\max} = \max_{0 \le \tau \le 1} x(\tau)$$

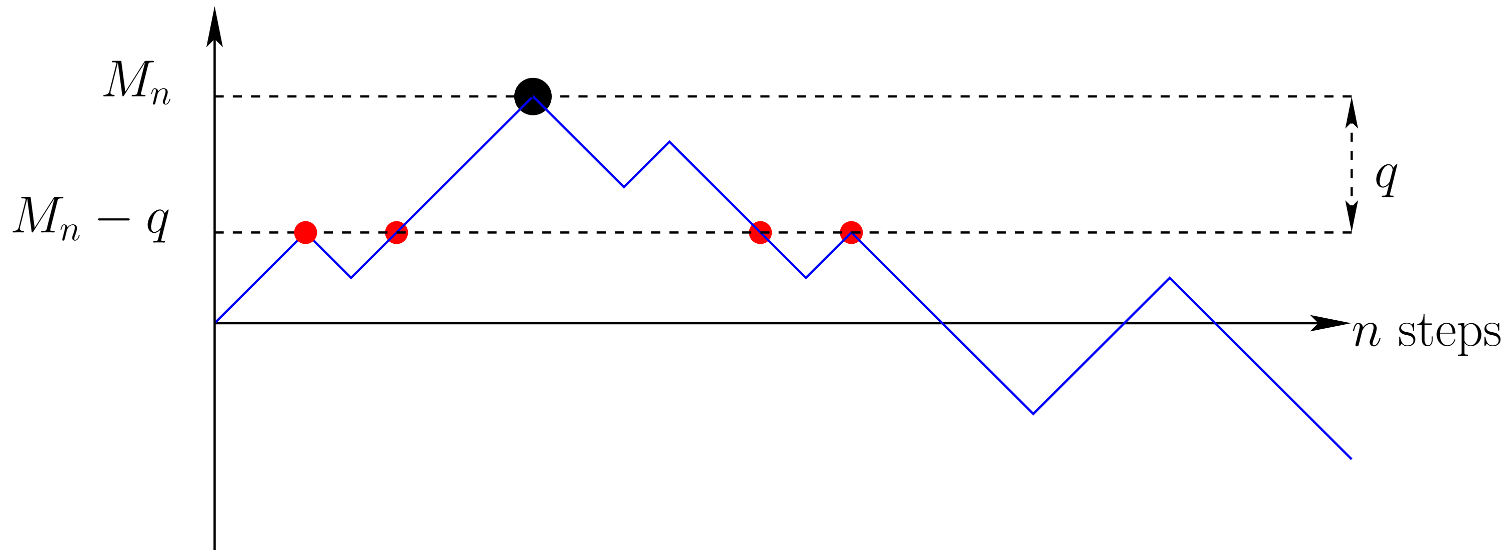
- Q: full statistics of the density of near-extremes $\rho(r)$?

$$\mathbb{E}[\rho^k(r)] = 8k! \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} [(2l+1)\Phi^{(k+1)}((2l+1)r) + (k-2(l+1))\Phi^{(k+1)}(2(l+1)r)]$$

$$\Phi^{(0)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \Phi^{(j+1)}(x) = \int_x^\infty \Phi^{(j)}(u)\, du$$

A. Perret, A. Comtet, S. N. Majumdar, G. S. '13

Motivations



$$D(q,n) = \#\{0 \leq k \leq n \,|\, X_k = M_n - q\}$$

- ``Crowding'' near the maximum: is the maximum lonely at the top ?

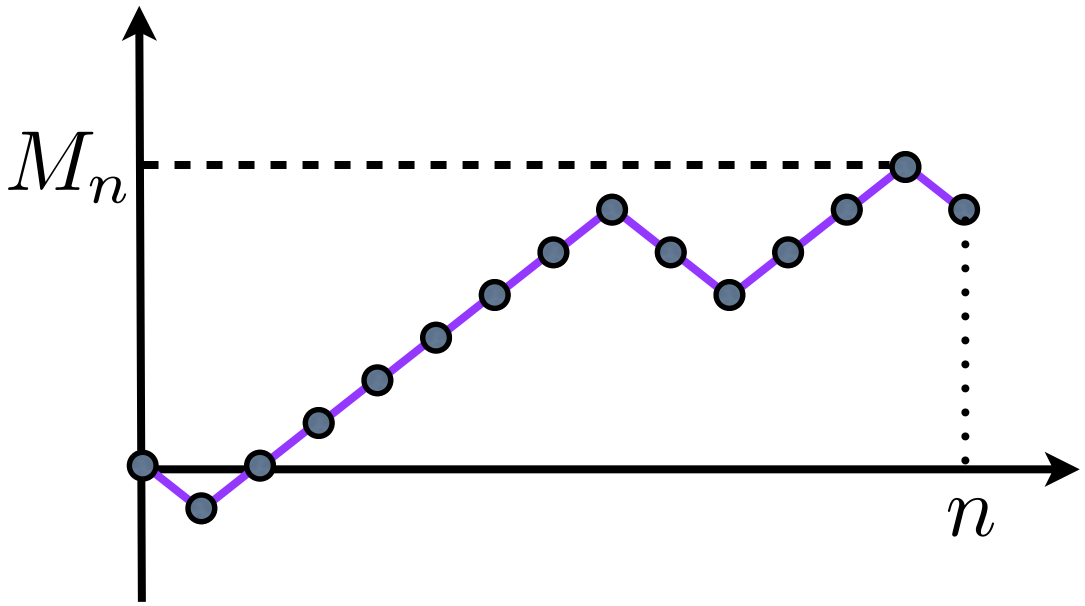
Sabhapandit, Majumdar '07

- Plays an important role in the analysis of the optimal algorithm to find the maximum of a random walk

Odlyzko '95
Hwang '97, Chassaing '99
Chassaing, Marckert, Yor '99

- Applications to ``functionals'' of the RW and Brownian motion

Search algorithm for the maximum of a RW



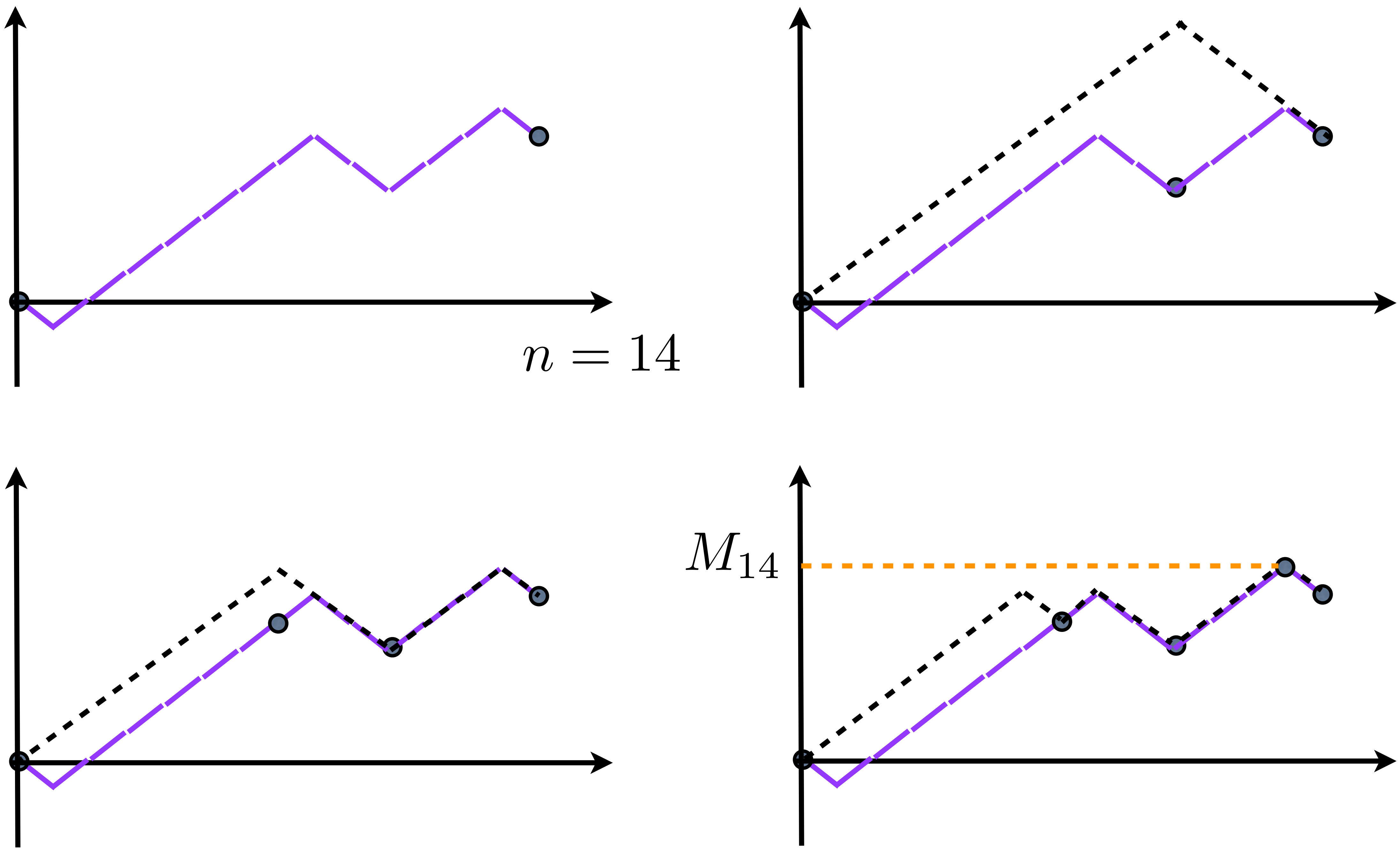
- $a \in A_n$: algorithm that finds M_n
- Cost of the algorithm:

  $C(a)$ = number of probes needed
- Q: what is the optimal algorithm ?

- The simplest algo. probes all the positions: its cost is n

- Because of the correlations between the positions of the random walker, one can usually do much better

Searching for the maximum of a RW: exploiting correlations



The maximum is found in 4 probes (4 < 14) !

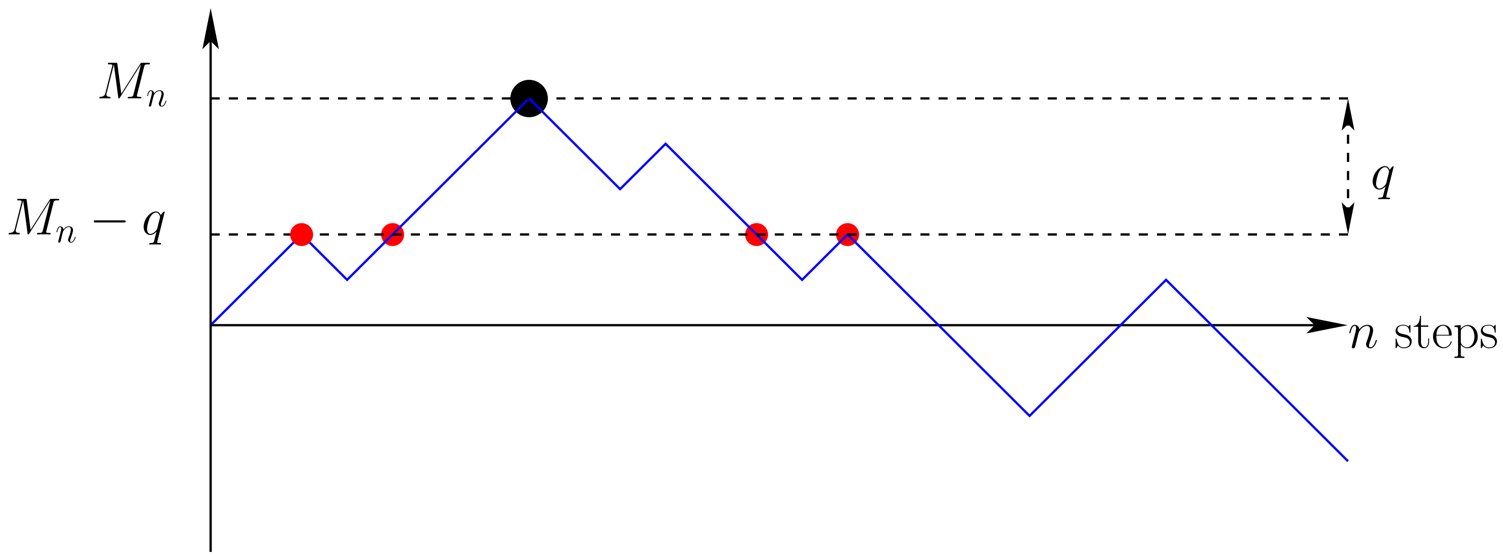
Searching for the maximum of a RW: optimal algorithm

- Average case optimality

$$\min_{a \in A_n} \mathbb{E}(C(a)) = c_0\sqrt{n} + o(\sqrt{n})$$

Odlyzko '95

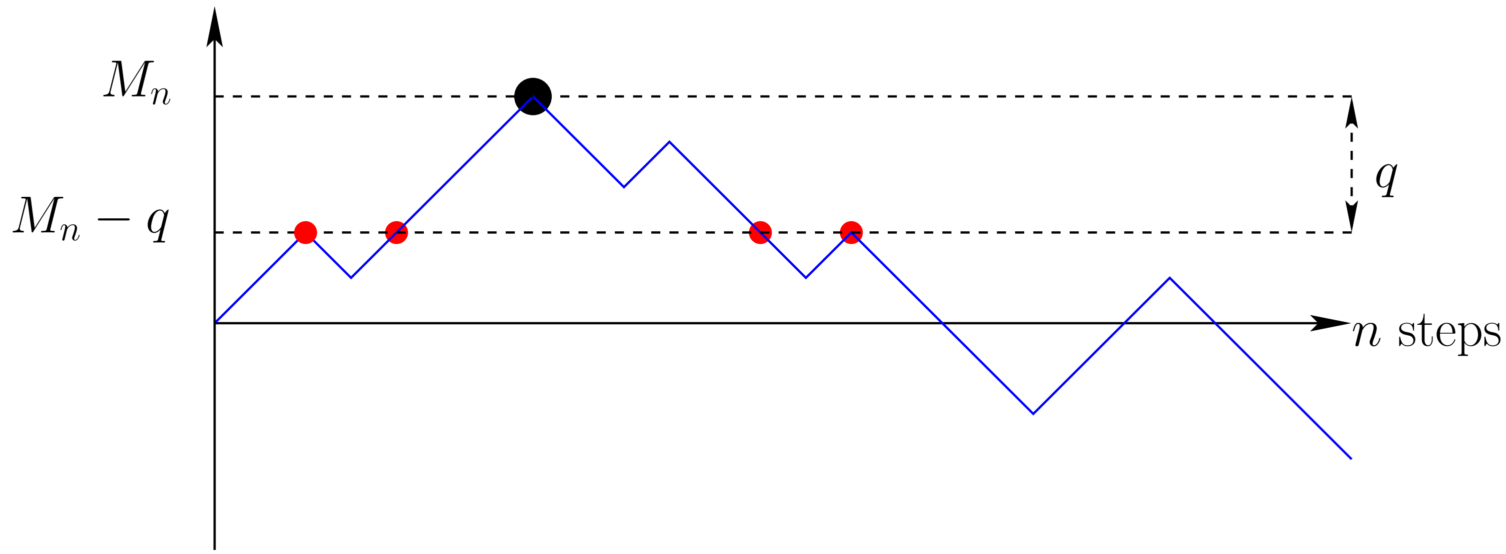
" In particular we need to prove that random walks do not spend much time close to their maxima."



$$c_0 = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} \sum_{q=0}^{n} \frac{\mathbb{E}[D(q,n)]}{q+1}$$

$$D(q,n) = \#\{0 \le k \le n \,|\, X_k = M_n - q\}$$

Motivations



$$D(q,n) = \#\{0 \le k \le n \,|\, X_k = M_n - q\}$$

- ☑ \`\`Crowding" near the maximum: is the maximum lonely at the top ?

Sabhapandit, Majumdar '07

- ☑ Plays an important role in the analysis of the optimal algorithm to find the maximum of a random walk

Odlyzko '95
Hwang '97, Chassaing '99
Chassaing, Marckert, Yor '99

- Applications to \`\`functionals" of the RW and Brownian motion

Searching for the maximum of a RW: optimal algorithm

- Average case optimality

$$\min_{a\in A_n} \mathbb{E}(C(a)) = c_0\sqrt{n} + o(\sqrt{n})$$

Odlyzko '95

$$c_0 = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{dy}{y} \int_0^1 \frac{1}{\sqrt{w}} \exp\left(-\frac{y^2}{2w}\right) \operatorname{erf}\left(\frac{y}{\sqrt{2(1-w)}}\right) dw$$

after some manipulations...

$$c_0 = \sqrt{\frac{8}{\pi}} \log 2$$

Hwang '97, Chassaing '99

- Connection with a functional of the maximum of Brownian motion

Chassaing '99

Chassaing, Marckert, Yor '99

$x(\tau)$: Brownian motion

$$x_{\max} = \max_{0\leq\tau\leq1} x(\tau)$$

$$\boxed{c_0 = \mathbb{E}(I)\ ,\ I = \frac{1}{2}\int_0^1 \frac{d\tau}{x_{\max} - x(\tau)}}$$

Searching for the maximum of a RW: optimal algorithm

$$\min_{a \in A_n} \mathbb{E}(C(a)) = c_0 \sqrt{n} + o(\sqrt{n})$$ Odlyzko '95

$$\boxed{c_0 = \mathbb{E}(I) \; , \; I = \frac{1}{2} \int_0^1 \frac{d\tau}{x_{\max} - x(\tau)}} \qquad c_0 = \sqrt{\frac{8}{\pi}} \log 2$$

- Odlyzko described an algorithm $\mathrm{Od}(n)$ which is quasi-optimal

$$\mathbb{E}[C(\mathrm{Od}(n))] = c_0 \sqrt{n} + o(\sqrt{n})$$

- $\mathrm{Od}(n)$ is quasi-optimal in distribution (not only on average) and it was shown that

Chassaing, Marckert, Yor '99

$$\lim_{n \to \infty} \Pr\left(\frac{C(\mathrm{Od(n)})}{\sqrt{n}} \geq x\right) = \Pr(I \geq x)$$

→ relevance of a functional of the maximum of Browian motion

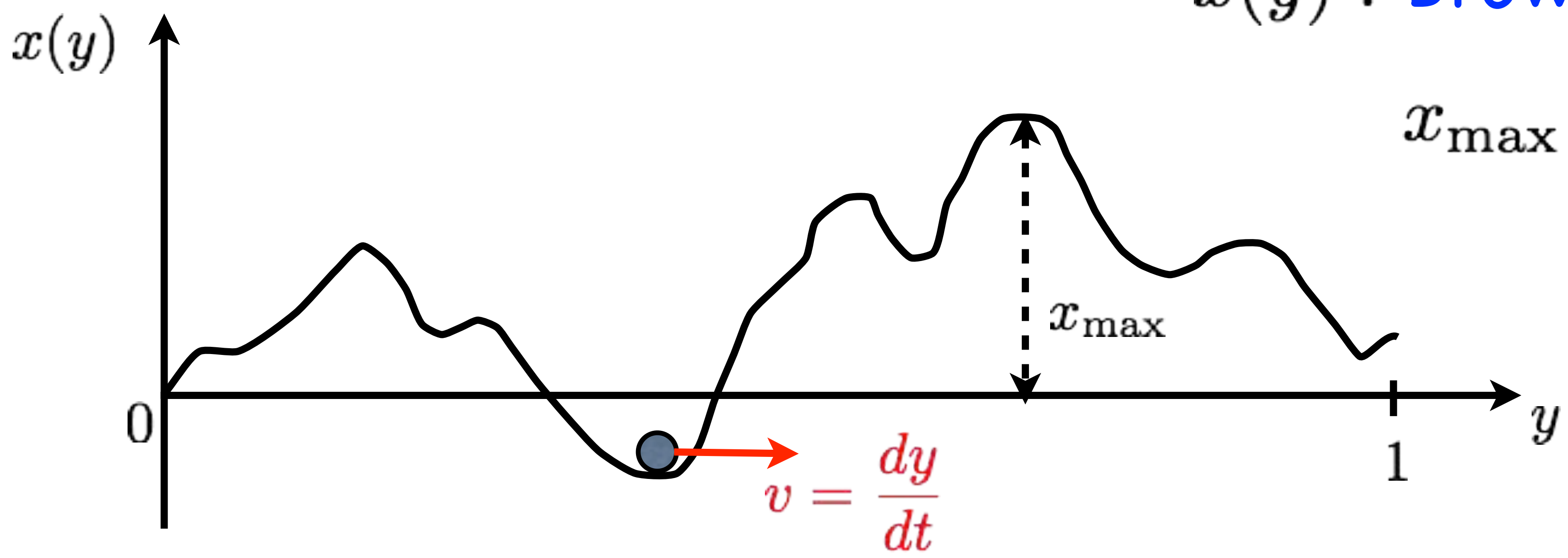
$$\boxed{I = \int_0^1 V(x_{\max} - x(\tau)) d\tau \; , \; V(x) = \frac{1}{2\,x}}$$

Functionals of the maximum of BM in physics

- Largest exit time of classical particles moving ballistically through a disordered Brownian potential

$x(y)$: Brownian motion

$$x_{\max} = \max_{0 \le y \le 1} x(y)$$



the slowest particle that crosses the sample is such that

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + x(y) = x_{\max}$$

and the largest time to cross the sample is $\boxed{T_{\max} = \frac{1}{\sqrt{2}} \int_0^1 \frac{dy}{\sqrt{x_{\max} - x(y)}}}$

An interesting family of functionals of the maximum of BM

$$\boxed{T_\alpha(t) = \int_0^t (x_{\max} - x(\tau))^\alpha \, d\tau}$$

- For $\alpha = -1$ it describes the cost of Odlyzko's algorithm
- For $\alpha = -\frac{1}{2}$ it describes the largest time to cross a Brownian barrier
- For $\alpha = +1$ it describes an area or ``Airy'' type of random variable

→ In this work we develop tools to study the statistics of such functionals of the maximum of BM

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013 & 2014

Outline

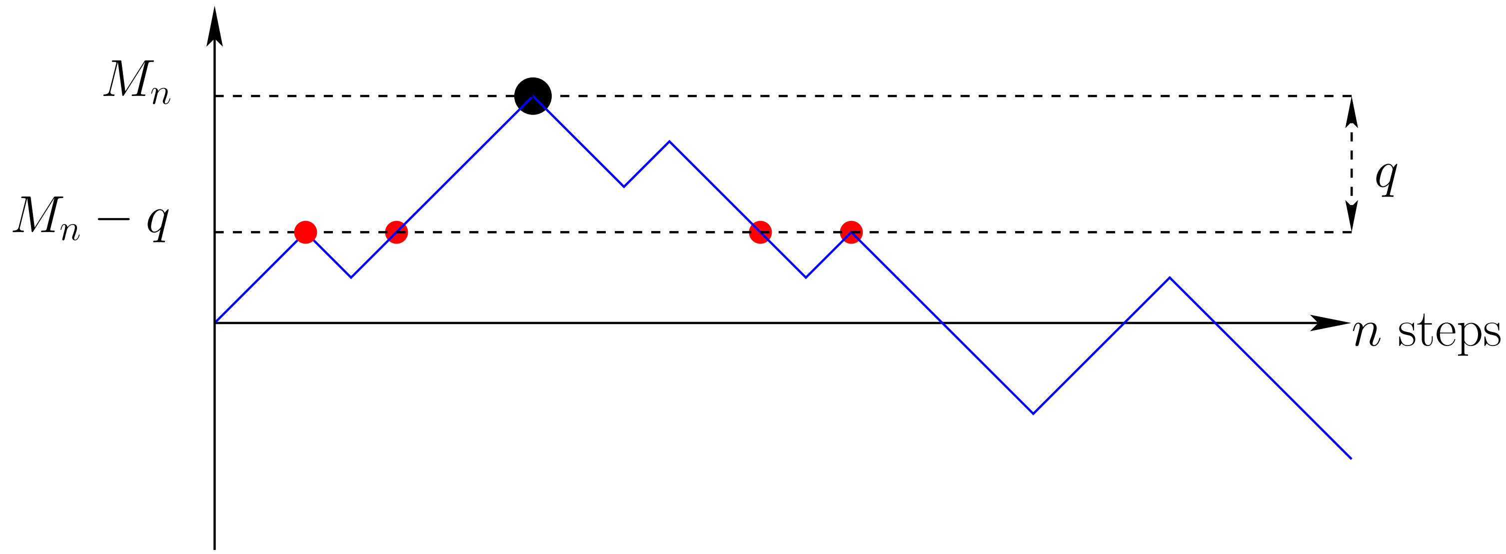
- Path counting method (based on propagators of BM)

- Feynman-Kac approach

- Applications of the Feynman-Kac approach

- Conclusion

Average number of near-extremal points for RW



$$D(q,n) = \#\{0 \le k \le n \,|\, X_k = M_n - q\}$$

- Odlyzko obtained an exact formula for $\mathbb{E}(D(q,n))$ Odlyzko '95

$$\mathbb{E}(D(q,n)) = \sum_{m=0}^{n} (A(n,m,q) + B(n,m,q))$$

$$A(n,m,q) = 2^{-n} \binom{m}{\lfloor \frac{m+q+1}{2} \rfloor} \sum_{j=0}^{q} \binom{n-m}{\lfloor \frac{n-m+j}{2} \rfloor}$$

$$B(b,m,q) = 2^{-n} \binom{m}{\lfloor \frac{n-m-q}{2} \rfloor} \sum_{j=0}^{q-1} \binom{m}{\lfloor \frac{m-q+1}{2} \rfloor + j}$$

What about the asymptotic limit $n \to \infty$?

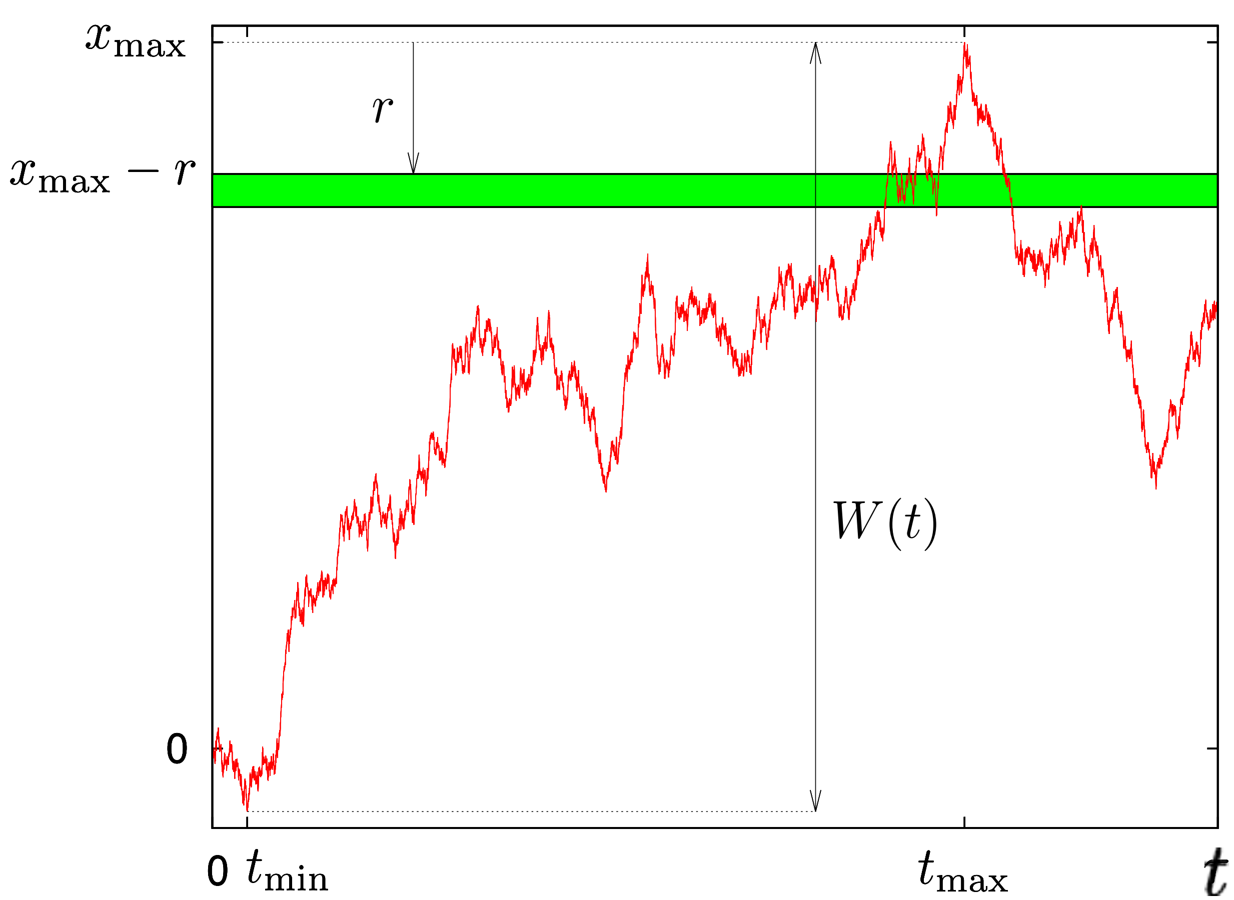
Density of near-extremes for Brownian motion

- Asymptotic limit $n \to \infty$

$$\frac{1}{\sqrt{n}} D(q = \lfloor r\sqrt{n} \rfloor, n) \xrightarrow[\text{law}]{} \rho(r) = \int_0^1 \delta(x_{\max} - x(\tau) - r) d\tau$$

$x(\tau)$: Brownian motion

$$x_{\max} = \max_{0 \leq \tau \leq 1} x(\tau)$$



$\rho(r)dr$: time spent by the BM in $[x_{\max} - r - dr, x_{\max} - r]$

$$\int_0^\infty \rho(r) dr = 1$$

$$\boxed{\mathbb{E}[\rho(r)] = \ ?}$$

Average density of near-extremes for Brownian motion

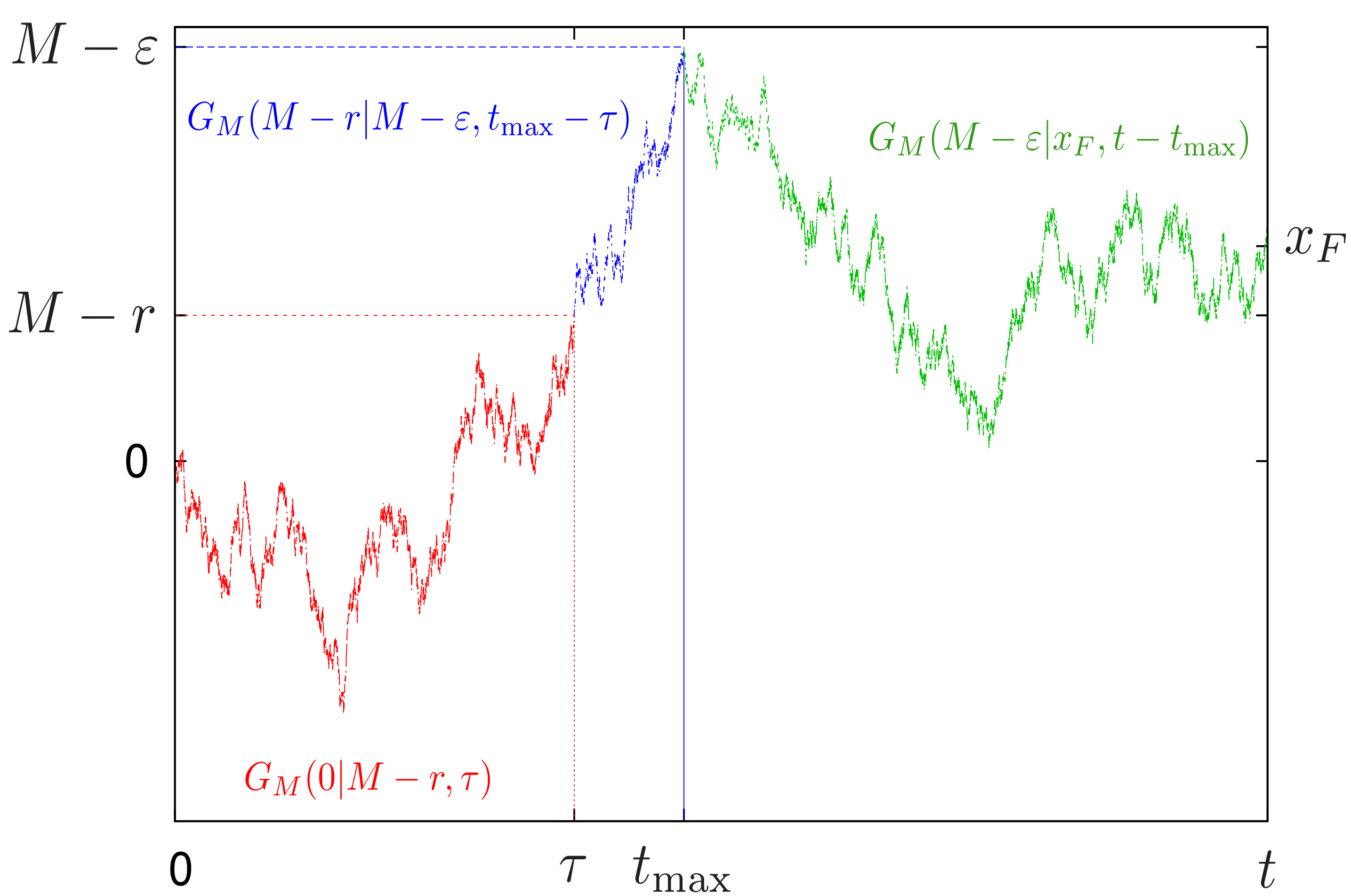
- Propagator of Brownian motion

$$G_M(\alpha|\beta,t)d\beta = \text{Pr}.\left[x(t)\in[\beta,\beta+d\beta]\;\middle|\;x(0)=\alpha\;\;\&\;\;x(\tau)<M\;\;\forall\tau\in[0,t]\right]$$

$$G_M(\alpha|\beta,t) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(\beta-\alpha)^2}{2t}} - e^{-\frac{(2M-\beta-\alpha)^2}{2t}}\right)$$

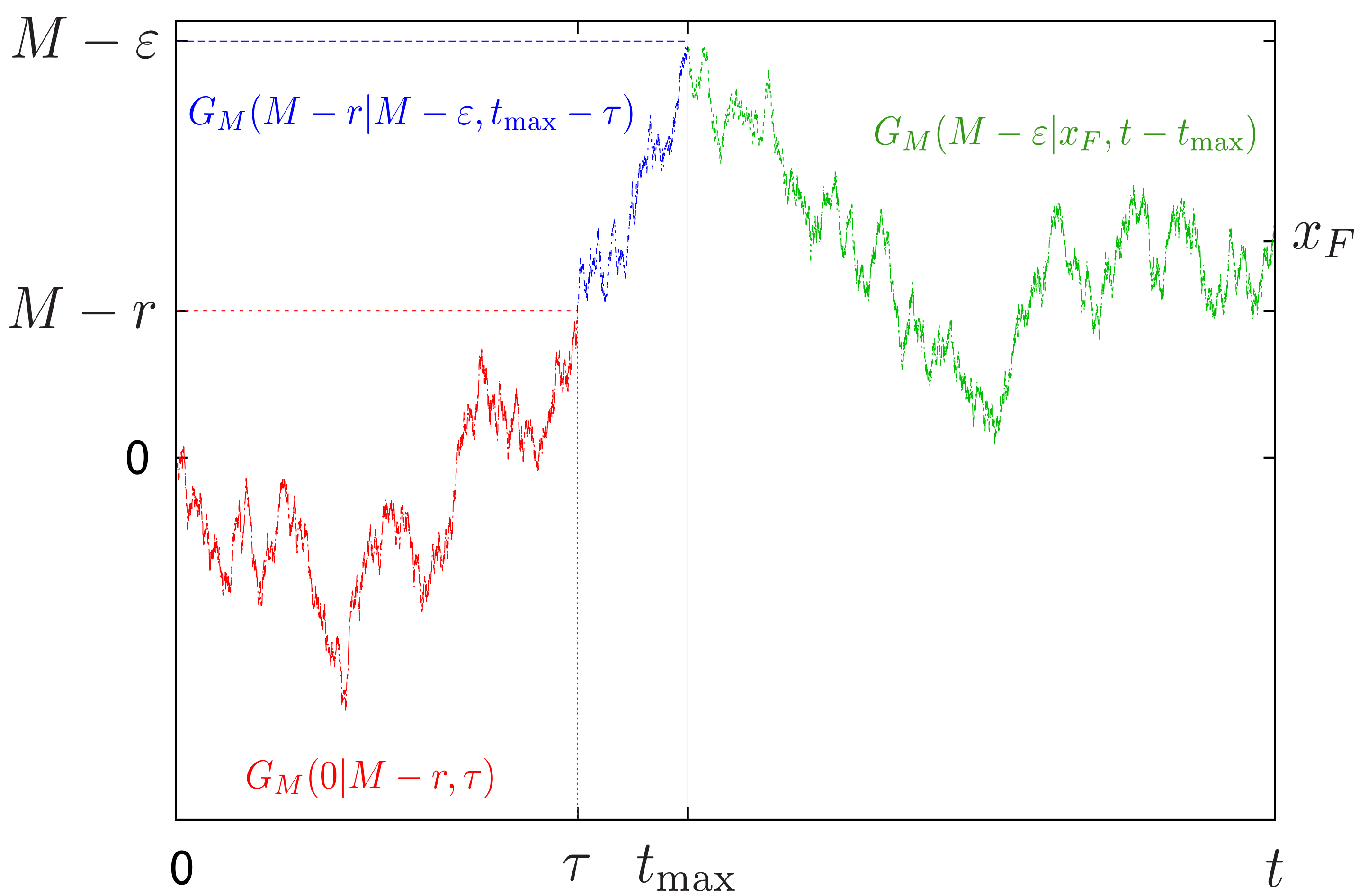
- Average density of near-extremes: counting paths method

$$\mathbb{E}[\rho(r,t)] = \int_0^t \mathbb{E}[\delta(x_{\max}-x(\tau)-r)]d\tau$$



Average density of near-extreme: counting paths

$$\mathbb{E}[\rho(r,t)] = \int_0^t \mathbb{E}[\delta(x_{\max} - x(\tau) - r)]d\tau$$



$$\mathbb{E}[\rho(r,t)] = \lim_{\varepsilon\to 0}\frac{2}{Z(\varepsilon)}\int_0^\infty dM \int_0^t dt_{\max}\int_{-\infty}^M dx_F \int_0^{t_{\max}} d\tau \quad G_M(0|M-r,\tau) G_M(M-r|M-\varepsilon, t_{\max}-\tau)$$
$$\times G_M(M-\varepsilon|x_F, t-t_{\max}),$$

$$Z(\varepsilon) = \int_0^\infty dM \int_0^t dt_{\max}\int_{-\infty}^M dx_F \, G_M(0|M-\varepsilon, t_{\max})\, G_M(M-\varepsilon|x_F, t-t_{\max})$$

Average density of near-extremes: counting paths

$$\mathbb{E}[\rho(r,t)] = \int_0^t \mathbb{E}[\delta(x_{\max} - x(\tau) - r)] d\tau$$

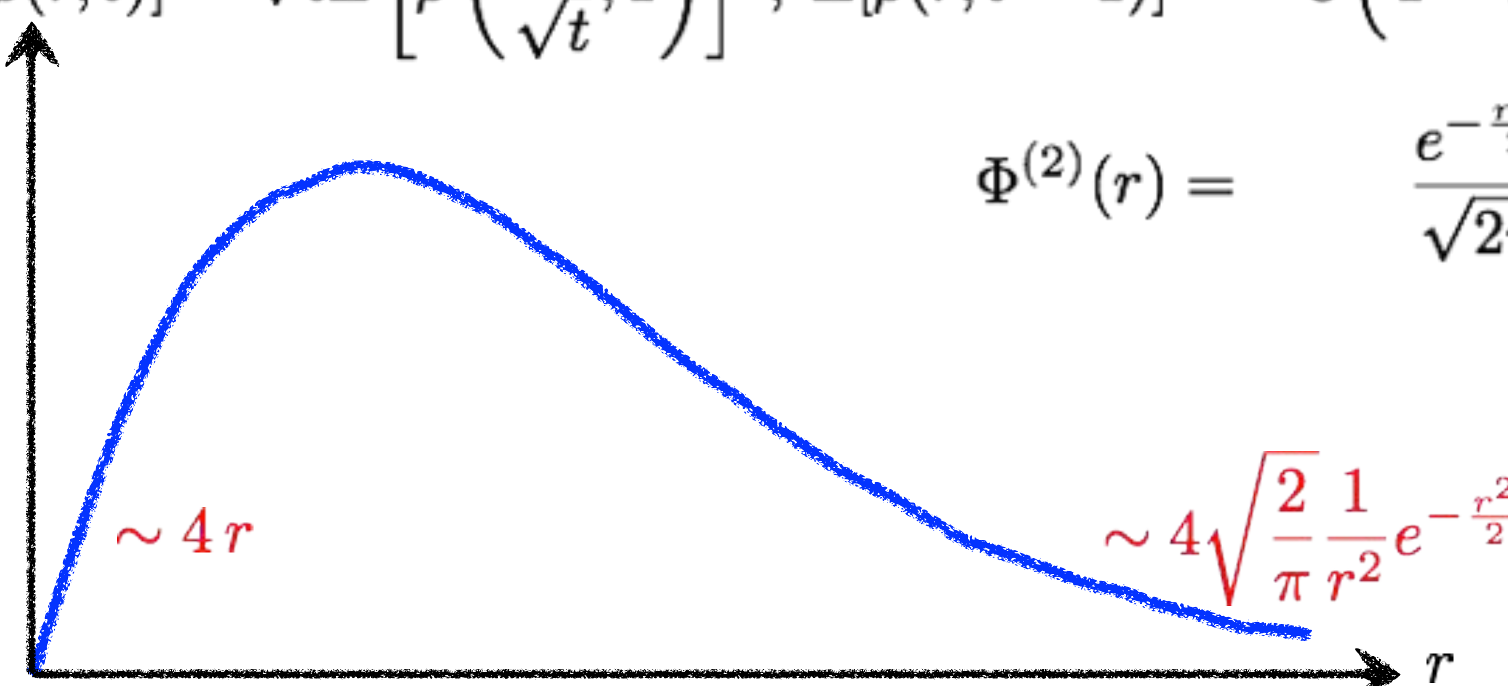
- Use of Laplace transform with respect to time t

$$\int_0^\infty dt e^{-st} \mathbb{E}[\rho(r,t)] = 8 \frac{e^{-\sqrt{2s}r} - e^{-2\sqrt{2s}r}}{(2s)^{3/2}}$$

- And finally...

$$\mathbb{E}[\rho(r,t)] = \sqrt{t}\,\mathbb{E}\left[\rho\left(\frac{r}{\sqrt{t}}, 1\right)\right] \ , \ \mathbb{E}[\rho(r, t=1)] = \ 8\left(\Phi^{(2)}(r) - \Phi^{(2)}(2r)\right) \ ,$$

$$\Phi^{(2)}(r) = \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}} - \frac{r}{2}\mathrm{erfc}\left(\frac{r}{\sqrt{2}}\right)$$



Application to the average cost of Odlyzko 's algorithm

- Average cost of the optimal algorithm for the search of the maximum of RW

$$\mathbb{E}[C(\mathrm{Od}(n))] = c_0\sqrt{n} + o(\sqrt{n})$$

$$c_0 = \mathbb{E}(I)\ ,\ I = \frac{1}{2}\int_0^1 \frac{d\tau}{x_{\max} - x(\tau)}$$

- Using the average density of near-extremes of Brownian motion

$$\mathbb{E}[\rho(r)] = \int_0^1 \mathbb{E}[\delta(x_{\max} - x(\tau) - r)]d\tau$$

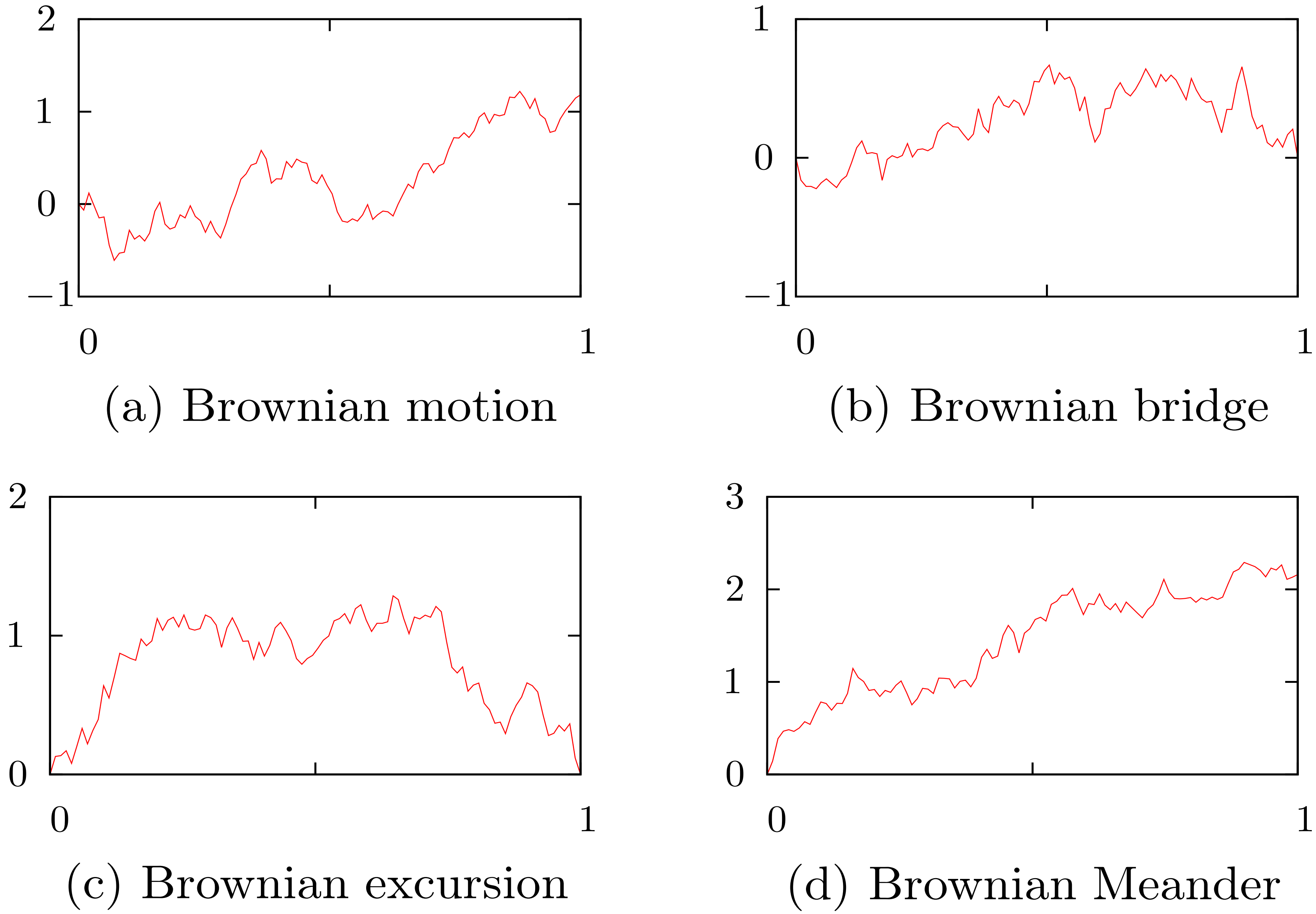
$$\Longrightarrow \boxed{c_0 = \frac{1}{2}\int_0^\infty \frac{\mathbb{E}[\rho(r)]}{r}dr = \sqrt{\frac{8}{\pi}}\log 2}$$

- For more general functionals of the maximum of BM

$$T_\alpha = \int_0^1 (x_{\max} - x(\tau))^\alpha d\tau\ ,\ \mathbb{E}[T_\alpha] = \int_0^\infty r^\alpha \rho(r)dr = \frac{(2-2^{-\alpha})\Gamma(\frac{1+\alpha}{2})}{(2+\alpha)\sqrt{\pi}}$$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013 & 2014

Average density of near-extremes of constrained BM

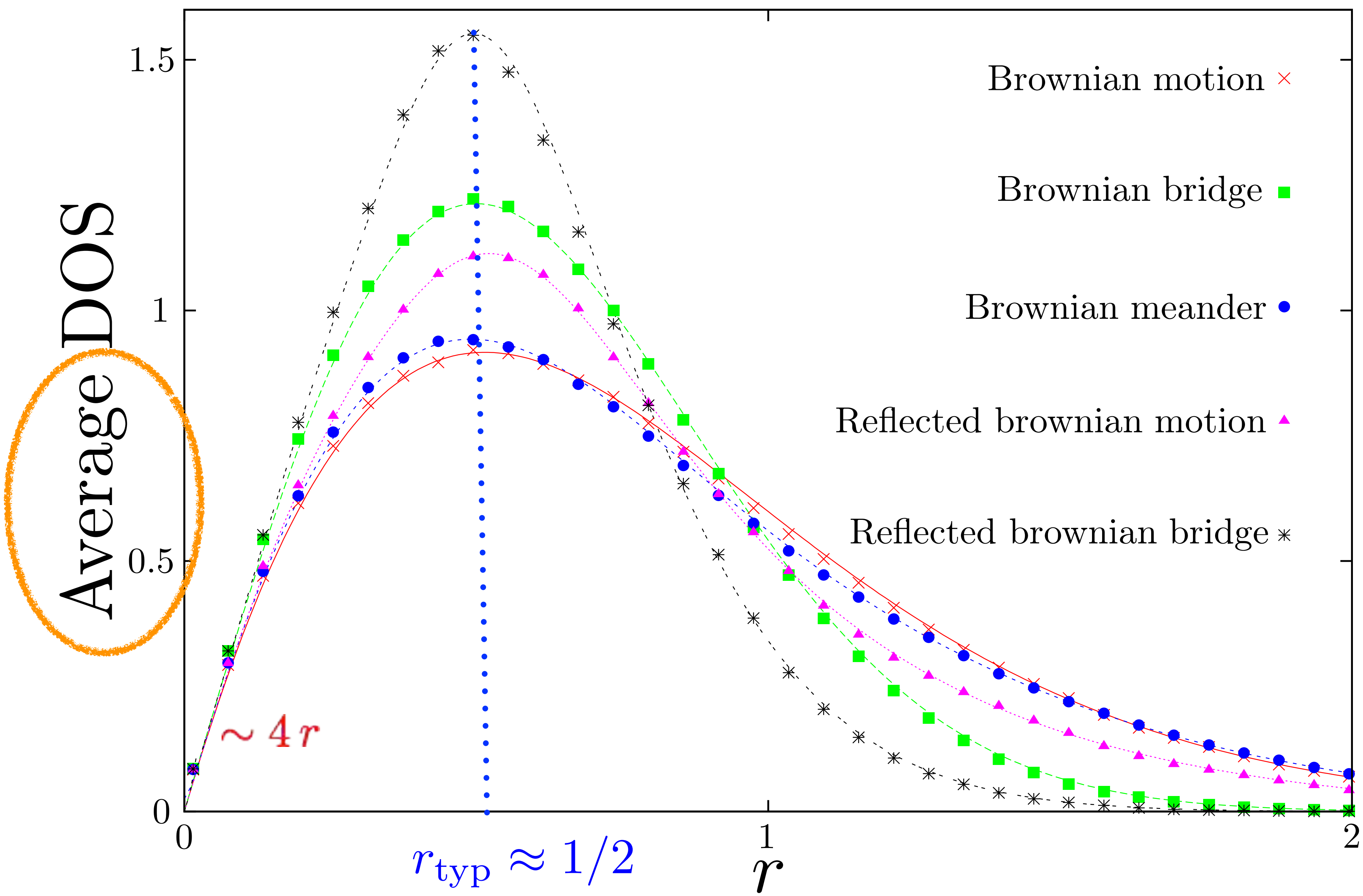


For instance, for the Brownian meander:

$$\mathbb{E}[\rho_{Me}(r)] = \sqrt{2\pi}\left(\sum_{n=1}^{\infty} \frac{4n(-1)^n}{2n^2+3(-1)^n-5}\operatorname{erfc}(nr) - \operatorname{erfc}(2r)\right)$$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013 & 2014

Average density of near-extremes of constrained BM



A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013 & 2014

What about higher moments of such functionals ?

Outline

- ☑ Path counting method (based on propagators of BM)

- ☐ Feynman-Kac approach

- ☐ Applications of the Feynman-Kac approach

- ☐ Conclusion

Exponential functionals of the maximum of BM

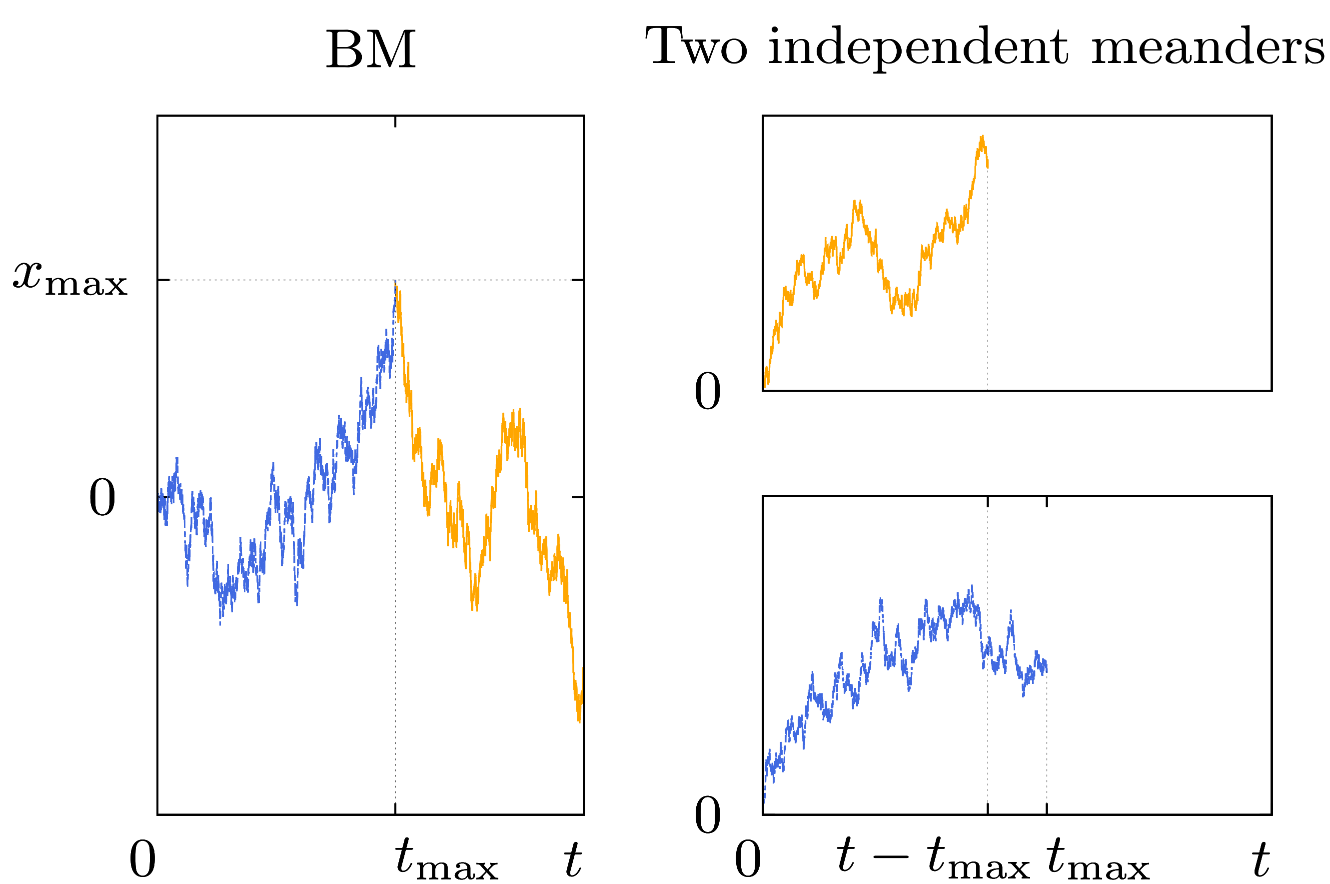
- Functional of the maximum of Brownian motion :

$$\mathcal{O}_{\max}(t) = \int_0^t V(x_{\max} - x(\tau))d\tau$$

- Goal: compute the Laplace transform of the PDF of $\mathcal{O}_{\max}(t)$:

$$\mathbb{E}\left[\exp\left(-\lambda\int_0^t V(x_{\max} - x(\tau))d\tau\right)\right]$$

- Decompose the path into two independent meanders

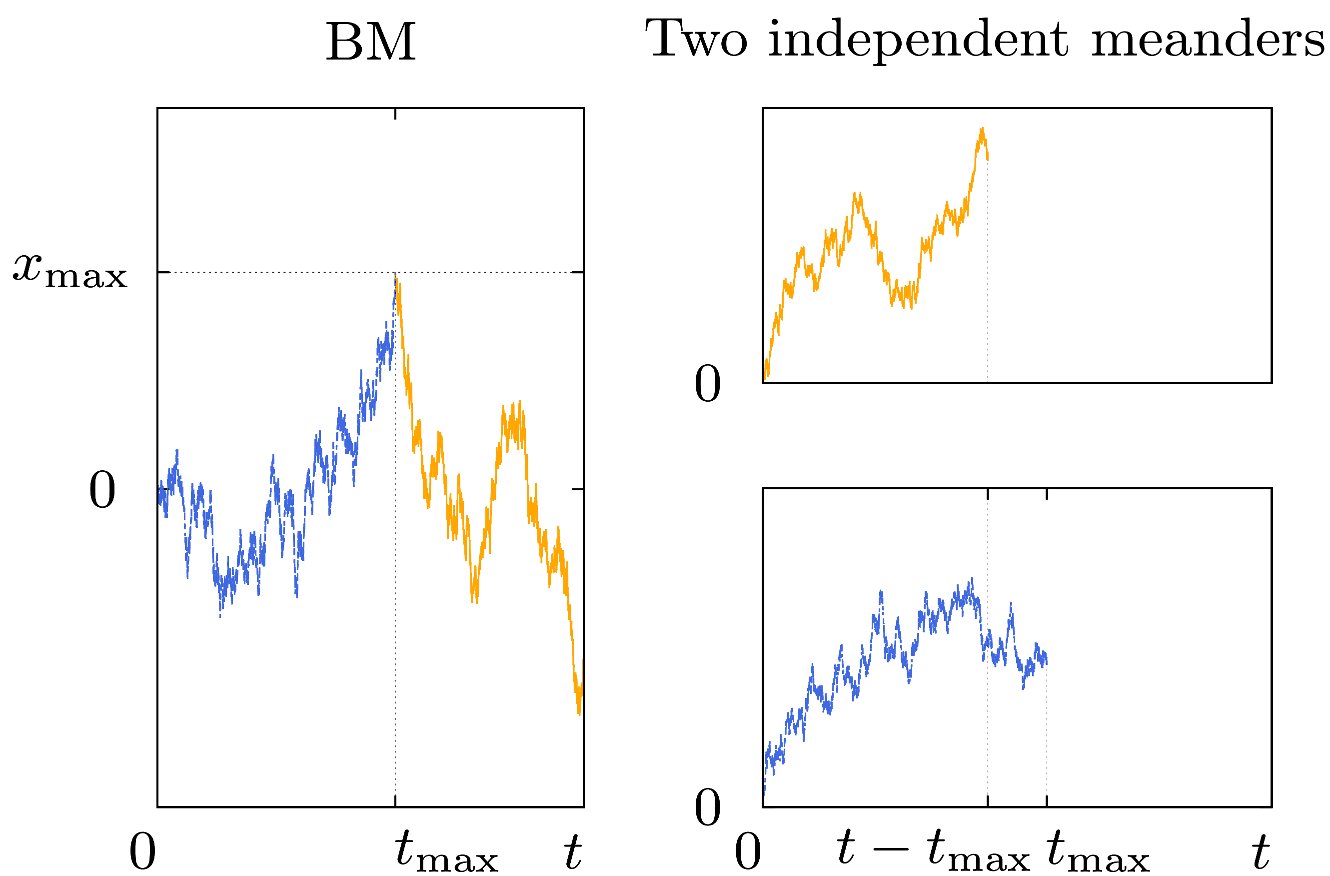


$$\mathrm{Pr.}\,(t_{\max} \leq T) = \int_0^T \frac{dx}{\pi\sqrt{x(t-x)}}$$

Lévy 's arcsine law

Reducing to functionals of the Brownian meander

- Decompose the path into two independent meanders



$$\Pr.\,(t_{\max} \leq T) = \int_0^T \frac{dx}{\pi\sqrt{x(t-x)}}$$

Lévy 's arcsine law

$$\mathbb{E}[e^{-\lambda \int_0^t d\tau V(x_{\max} - x(\tau))}] = \int_0^t dt_{\max} \varphi(t_{\max}) \varphi(t - t_{\max})$$

$$\varphi(\tau) = \frac{1}{\sqrt{\pi \tau}} \mathbb{E}_+ \left[\exp\left(-\lambda \int_0^\tau du\, V(x_{\mathrm{Me}}(u)) \right) \right]$$

$\Longrightarrow$ Back to functionals of the Brownian meander

Feynman-Kac formula

$$\mathbb{E}[e^{-\lambda \int_0^t d\tau V(x_{\max}-x(\tau))}] = \int_0^t dt_{\max} \varphi(t_{\max})\varphi(t-t_{\max})$$

$$\varphi(\tau) = \frac{1}{\sqrt{\pi\tau}} \mathbb{E}_+ \left[\exp\left(-\lambda \int_0^\tau du\, V(x_{\mathrm{Me}}(u))\right)\right]$$

- Laplace transform with respect to time t

$$\boxed{\begin{aligned} &\int_0^\infty e^{-st}\mathbb{E}\left(e^{-\lambda \int_0^t V(x_{\max}-x(\tau))d\tau}\right) dt = [\tilde{\varphi}(s)]^2 \ , \\ &\tilde{\varphi}(s) = \int_0^\infty e^{-st}\varphi(t)\, dt \end{aligned}}$$

which can be computed as $\boxed{\tilde{\varphi}(s) = \frac{\sqrt{2}}{W}\int_0^\infty dy_F\, u_s'(0)v_s(y_F)}$

where $u_s(x), v_s(x)$ are two independent solutions of

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \lambda V(x) + s\right)\psi(x) = 0$$

such that $\lim_{x\to 0} u_s(x) = 0$ and $W = u_s'(x)v_s(x) - u_s(x)v_s'(x)$

$\lim_{x\to +\infty} v_s(x) = 0$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2014

Outline

- ☑ Path counting method (based on propagators of BM)
- ☑ Feynman-Kac approach
- ☐ Applications of the Feynman-Kac approach
- ☐ Conclusion

Feynman-Kac formula: application to the density of near-extremes

$$\mathcal{O}_{\max}(t) = \int_0^t V(x_{\max} - x(\tau))d\tau$$

- The density of near-extremes $\rho(r,t) = \displaystyle\int_0^t \delta(x_{\max} - x(\tau) - r)d\tau$

  corresponds to $V(x) = \delta(x - r)$

- Applying the general formalism to this specific case

Feynman-Kac formula

$$\mathbb{E}[e^{-\lambda\int_0^t d\tau V(x_{\max}-x(\tau))}] = \int_0^t dt_{\max}\varphi(t_{\max})\varphi(t-t_{\max})$$

$$\varphi(\tau) = \frac{1}{\sqrt{\pi\tau}}\mathbb{E}_+\left[\exp\left(-\lambda\int_0^\tau du\, V(x_{\mathrm{Me}}(u))\right)\right]$$

- Laplace transform with respect to time t

$$\boxed{\begin{aligned}&\int_0^\infty e^{-st}\mathbb{E}\left(e^{-\lambda\int_0^t V(x_{\max}-x(\tau))d\tau}\right)dt = [\tilde{\varphi}(s)]^2\,,\\ &\tilde{\varphi}(s) = \int_0^\infty e^{-st}\varphi(t)\,dt\end{aligned}}$$

which can be computed as $\boxed{\tilde{\varphi}(s) = \frac{\sqrt{2}}{W}\int_0^\infty dy_F\, u_s'(0)v_s(y_F)}$

where $u_s(x), v_s(x)$ are two independent solutions of

$$\left(-\frac{1}{2}\frac{d^2}{dx^2}+\lambda V(x)+s\right)\psi(x) = 0$$

such that $\lim_{x\to 0} u_s(x) = 0$ and $W = u_s'(x)v_s(x) - u_s(x)v_s'(x)$

$\lim_{x\to+\infty} v_s(x) = 0$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2014

Feynman-Kac formula: application to the density of near-extremes

$$\mathcal{O}_{\max}(t) = \int_0^t V(x_{\max} - x(\tau))d\tau$$

- The density of near-extremes $\rho(r,t) = \int_0^t \delta(x_{\max} - x(\tau) - r)d\tau$

  corresponds to $V(x) = \delta(x - r)$

- Applying the general formalism to this specific case

$$\int_0^\infty e^{-st}\mathbb{E}\left(e^{-\lambda\rho(r,t)}\right)dt = \frac{1}{s}\left(\frac{\sqrt{2s} + \lambda(1 - e^{-\sqrt{2s}r})^2}{\sqrt{2s} + \lambda(1 - e^{-2\sqrt{2s}r})}\right)^2$$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013

Feynman-Kac formula: applications to the density of near-extremes

From

$$\int_0^\infty e^{-st}\mathbb{E}\left(e^{-\lambda\rho(r,t)}\right)dt = \frac{1}{s}\left(\frac{\sqrt{2s}+\lambda(1-e^{-\sqrt{2s}r})^2}{\sqrt{2s}+\lambda(1-e^{-2\sqrt{2s}r})}\right)^2$$

we obtain the moments of arbitrary order

$$\mathbb{E}(\rho^k(r,t=1)) = 8k!\sum_{l=0}^{k-1}(-1)^l\binom{k-1}{l}[(2l+1)\Phi^{(k+1)}((2l+1)r)+(k-2(l+1))\Phi^{(k+1)}(2(l+1)r)]$$

where $\displaystyle \frac{e^{-\sqrt{2s}u}}{(\sqrt{2s})^{j+1}} = \int_0^\infty t^{\frac{j-1}{2}}\Phi^{(j)}\left(\frac{u}{\sqrt{t}}\right)e^{-st}dt$

These functions were studied in detail in Chassaing & Louchard '02

Feynman-Kac formula: applications to the cost of Odlyzko 's algorithm

$$\mathcal{O}_{\max}(t) = \int_0^t V(x_{\max} - x(\tau))d\tau$$

- The cost of Odlyzko 's algorithm is given by $I = \frac{1}{2}\int_0^t \frac{d\tau}{x_{\max} - x(\tau)}$

  which corresponds to $V(x) = \frac{1}{2x}$

- Applying the general formalism to this specific case

$$\int_0^\infty e^{-st}\mathbb{E}\left[e^{-\frac{\lambda}{2}\int_0^t \frac{d\tau}{x_{\max}-x(\tau)}}\right]dt = \frac{4}{s}\sum_{n=0}^{\infty}(-1)^n \frac{\lambda^n}{2^n(\sqrt{2s})^n}\sum_{k=0}^{n}\tilde{\zeta}(k)\tilde{\zeta}(n-k)$$

$$\tilde{\zeta}(k) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^k}$$

recovering, in a quite different way, the result of Chassaing, Marckert, Yor '99

Feynman-Kac formula: applications to more general functionals

$$T_\alpha(t) = \int_0^t (x_{\max} - x(\tau))^\alpha d\tau$$

corresponding to $V(x) = x^\alpha$

- Exact results for the second moment

$$\mathbb{E}(T_\alpha^2(t)) = \frac{t^{2+\alpha}}{2^{3\alpha}\Gamma(3+\alpha)} \left(\Gamma(\alpha+1)^2(2^\alpha - 1)(2^{\alpha+1} - 1) + \frac{\Gamma(3+2\alpha)(4^{\alpha+1} - 1)}{4(1+\alpha)^2} + \sum_{n=1}^{\infty} \frac{\Gamma(2+2\alpha+n)}{n! 2^{1+n}(1+\alpha+n)} \right)$$

A. Perret, A. Comtet, S. N. Majumdar, G. S., 2013

Outline

- ☑ Path counting method (based on propagators of BM)
- ☑ Feynman-Kac approach
- ☑ Applications of the Feynman-Kac approach
- ☐ Conclusion

Conclusion and perspectives

- Various tools to study the statistics of functionals of the maximum of Brownian motion

- Application to the full statistics of near-extremal points of long random walks

- Alternative method to study the cost of Odlyzko's optimal algorithm to find the maximum of long random walks

- Extension of these techniques to system with several random walkers ?

- What about more general stable processes ?