This presentation contains animations which require PDF browser which accepts JavaScript.

For best results use Acrobat Reader.

Asymptotic determinism of Robinson-Schensted-Knuth algorithm

joint work with Dan Romik

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and
Polska Akademia Nauk
and
Uniwersytet Wrocławski

Robinson-Schensted-Knuth algorithm — induction step

74	99		
23	53	69	
5	37	41	82

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41)$$

Robinson-Schensted-Knuth algorithm — induction step

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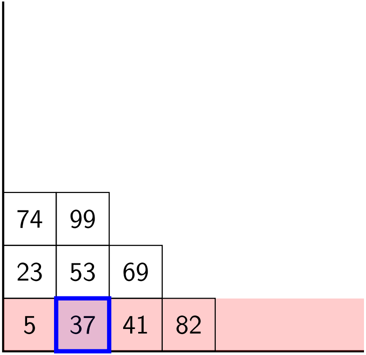

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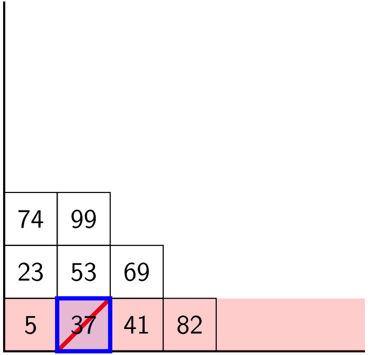

insertion tableau $P(\mathbf{x})$

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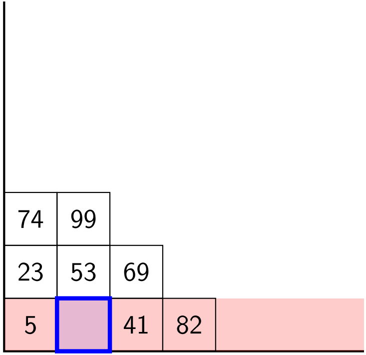

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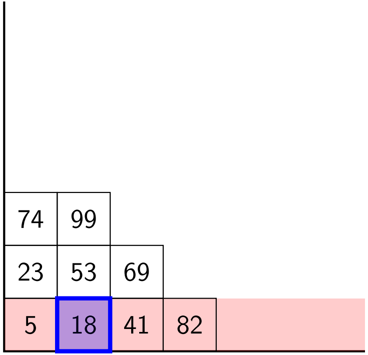

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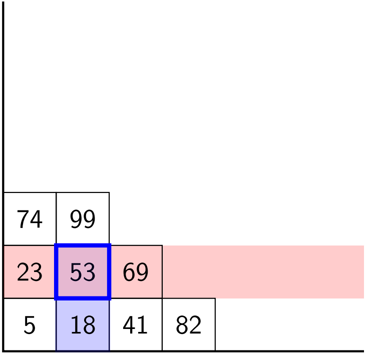

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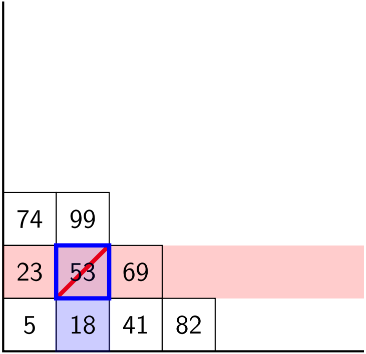

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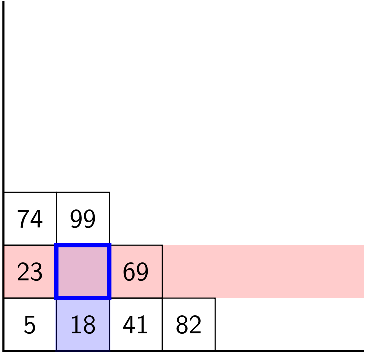

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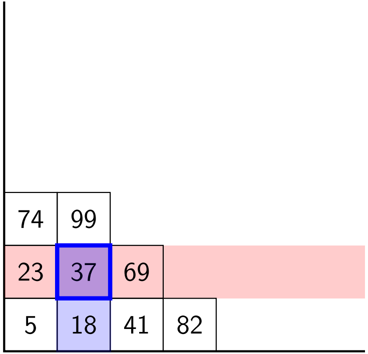

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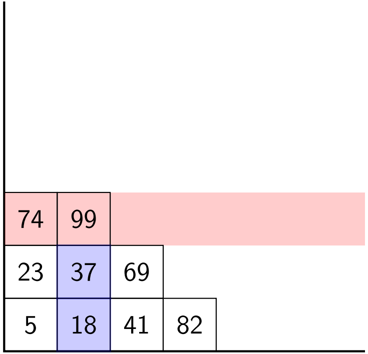

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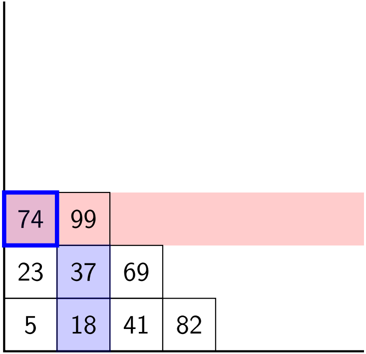

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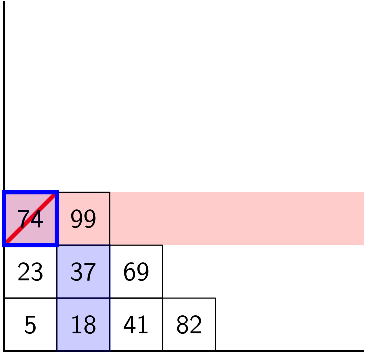

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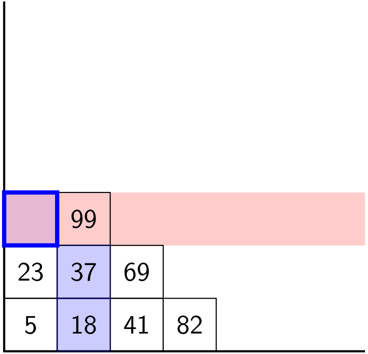

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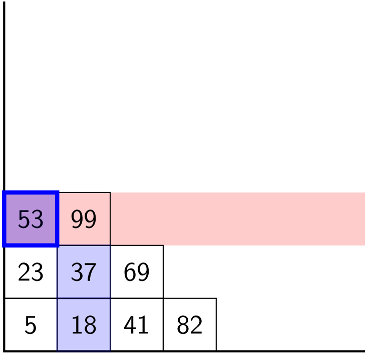
Robinson-Schensted-Knuth algorithm — induction step



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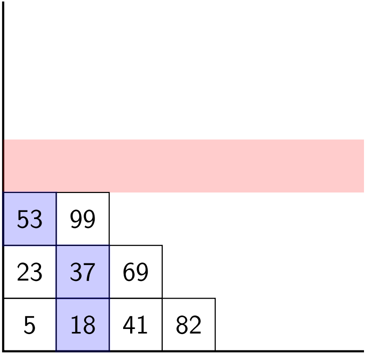

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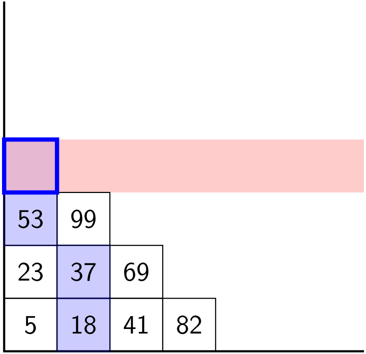

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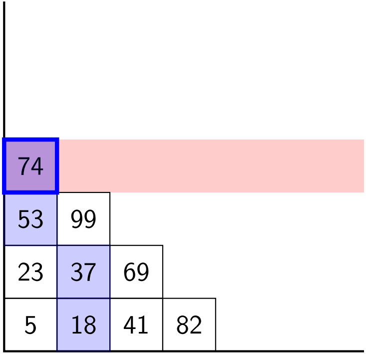

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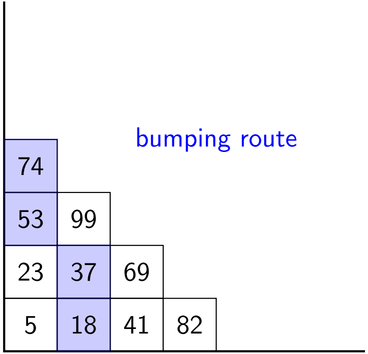
Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74			
53	99		
23	37	69	
5	18	41	82

insertion tableau $P(\mathbf{x})$

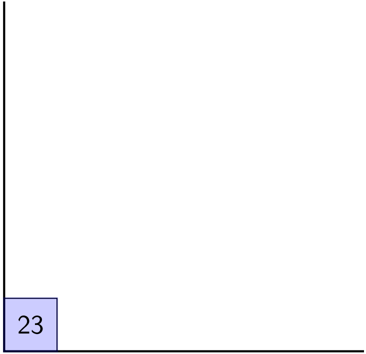
$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm

insertion tableau $P(\mathbf{x})$

$\mathbf{x} = \emptyset$

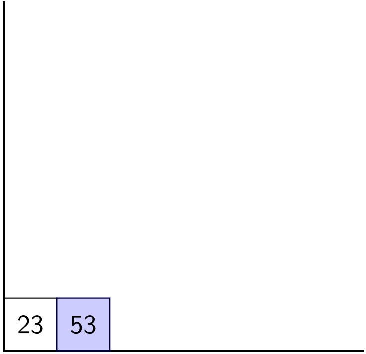
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, \ 53)$

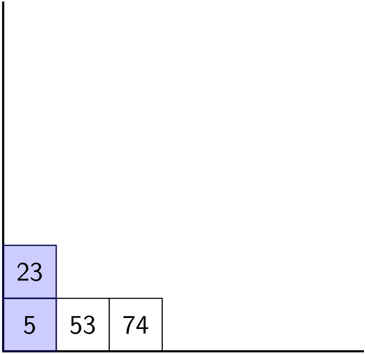
Robinson-Schensted-Knuth algorithm

23	53	74

insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74)$

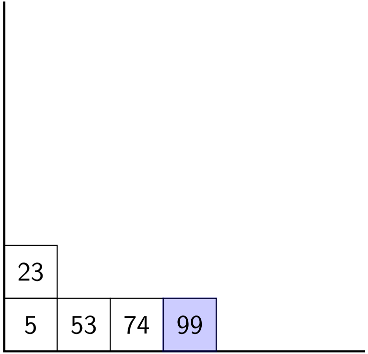
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23,\ 53,\ 74,\ 5)$$

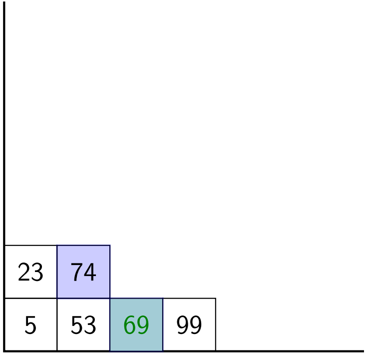
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99, 69)$

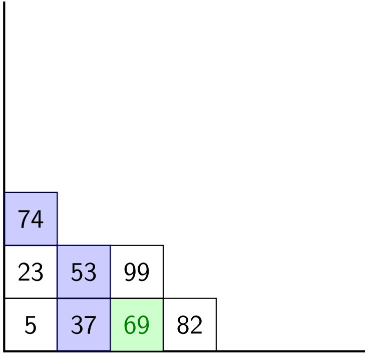
Robinson-Schensted-Knuth algorithm

23	74	99	
5	53	69	82

insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, {\color{green}69}, 82, {\color{blue}37})$$

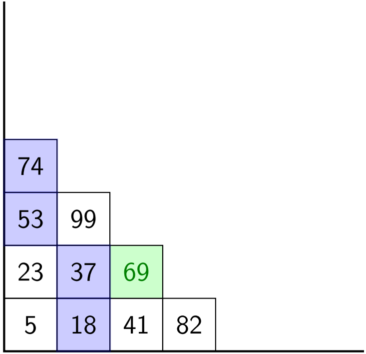
Robinson-Schensted-Knuth algorithm

74	99		
23	53	69	
5	37	41	82

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41)$$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm

74	99		
53	69		
23	37	41	
5	18	39	82

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18, 39)$$

Robinson-Schensted-Knuth algorithm

74	99		
53	69		
23	37	41	82
5	18	39	61

insertion tableau $P(\mathbf{x})$

$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18, 39, 61)$

Robinson-Schensted-Knuth algorithm

74	99			
53	69			
23	37	41	82	
5	18	39	61	73

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18, 39, 61, 73)$$

Robinson-Schensted-Knuth algorithm

74	99			
53	69	82		
23	37	41	73	
5	18	39	61	66

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18, 39, 61, 73, 66)$$

Robinson-Schensted-Knuth algorithm

74				
53	99			
41	69	82		
23	37	39	73	
5	18	22	61	66

insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23, 53, 74, 5, 99, 69, 82, 37, 41, 18, 39, 61, 73, 66, 22)$$

outlook

- $x_1, x_2, \ldots$ independent random variables with uniform distribution on the interval $[0, 1]$;
- insertion tableau $P_m = P(x_1, \ldots, x_m)$;

General problem

What can we say about (the time evolution of) the insertion tableau P_m?

"with the right scaling of time and space, the answer is deterministic (asymptotically)"

diffusion of a box

- $\boxed{x_n}(P_m)$ denotes the location of the box containing x_n in the insertion tableau P_m, for $m \geq n$;

Concrete problem 1

Suppose that n and x_n are known;
what can we say about the time evolution of $\boxed{x_n}(P_m)$
for $m = n, n+1, \ldots$ *?*

diffusion of a box

diffusion of a box

diffusion of a box

- $\boxed{x_n}(P_m)$ denotes the location of the box containing x_n in insertion tableau P_m, for $m \geq n$;

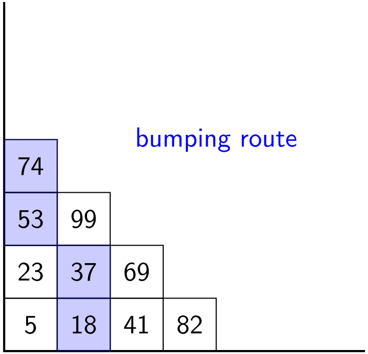
Theorem

There exists an explicit function $G : \mathbb{R}_+ \to \mathbb{R}_+^2$ such that

$$\frac{\boxed{x_n}(P_{\lfloor n e^{\tau} \rfloor})}{\sqrt{n\, x_n}} \xrightarrow[n\to\infty]{\text{in probability}} G_\tau \qquad \text{for } \tau \geq 0.$$

hydrodynamic limit of RSK

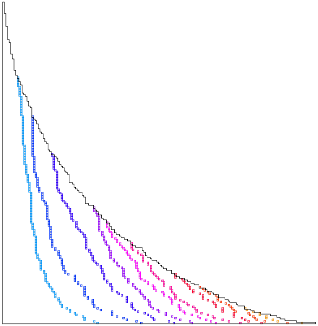
bumping routes

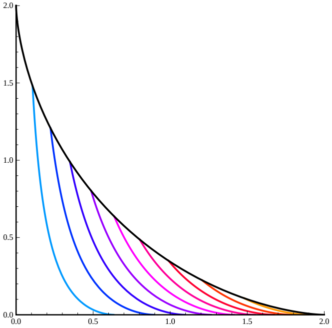


insertion tableau $P(\mathbf{x})$

$$\mathbf{x} = (23,\ 53,\ 74,\ 5,\ 99,\ 69,\ 82,\ 37,\ 41,\ \underbrace{18}_{x_n})$$

bumping routes



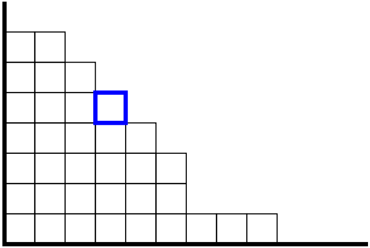


Theorem

Bumping route (scaled by factor $\frac{1}{\sqrt{n\ x_n}}$)
obtained by adding entry x_n to the tableau P_{n-1}
converges in probability (as $n \to \infty$) to a deterministic curve G_τ.

new box

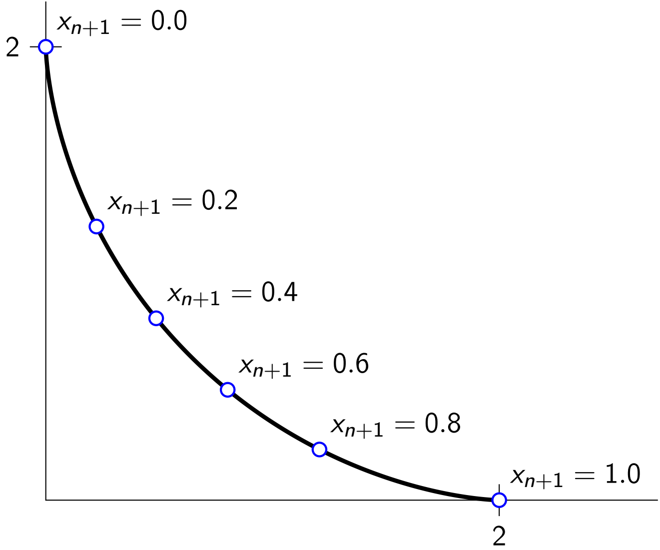
$$P(x_1,\ldots,x_n,x_{n+1})\setminus P(x_1,\ldots,x_n)=\{\square\}$$



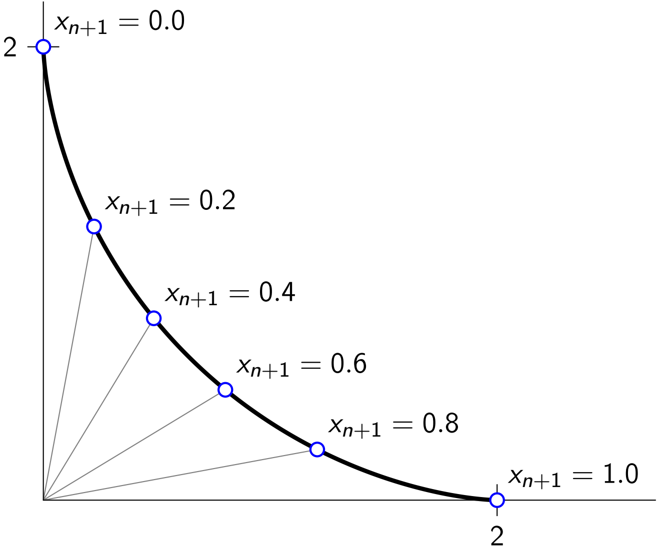
Theorem

$$\left\| \frac{\square}{\sqrt{n}} - \left(\mathrm{RSKcos}\,x_{n+1}, \mathrm{RSKsin}\,x_{n+1}\right)\right\| \xrightarrow[\textit{in probability}]{n\to\infty} 0$$

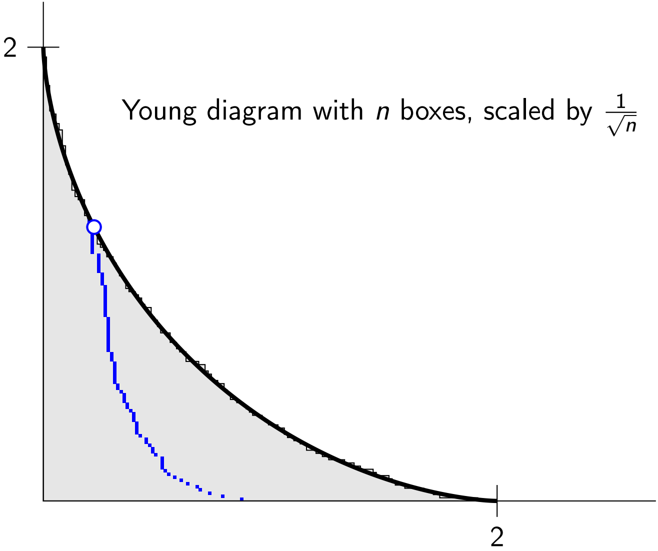
new box



new box



the key result explains the behavior of bumping routes



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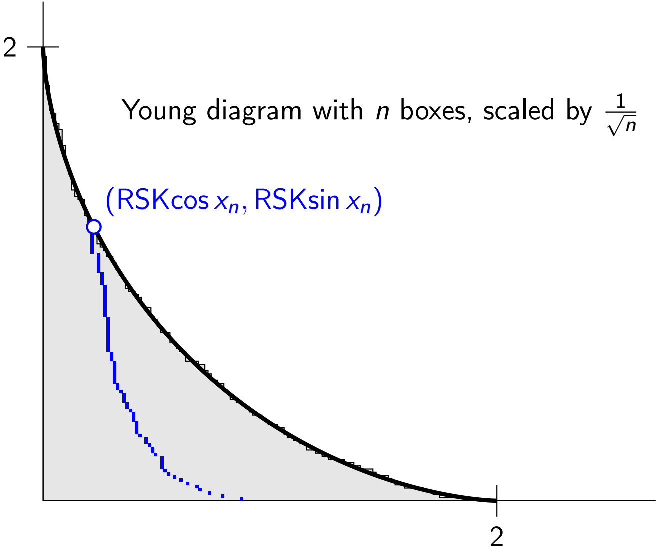

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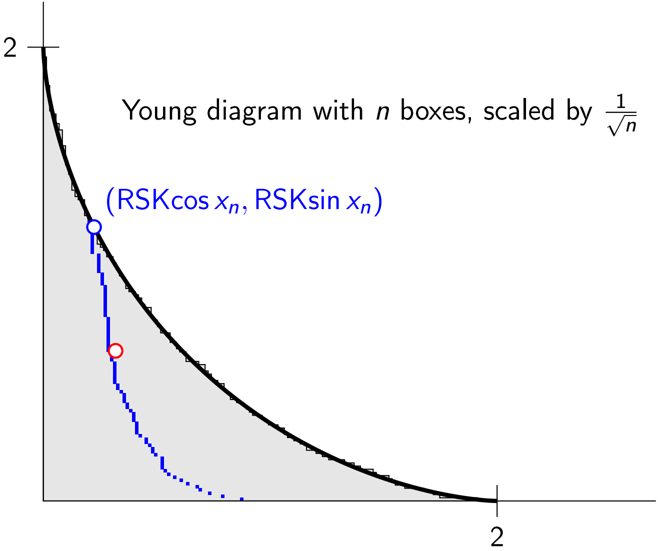

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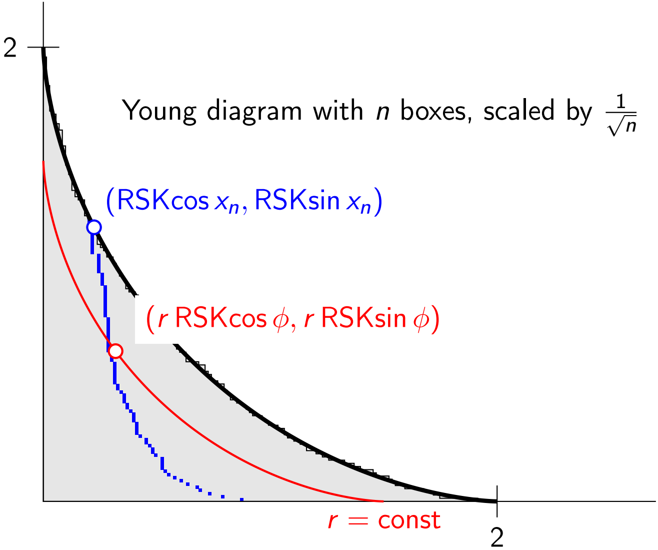

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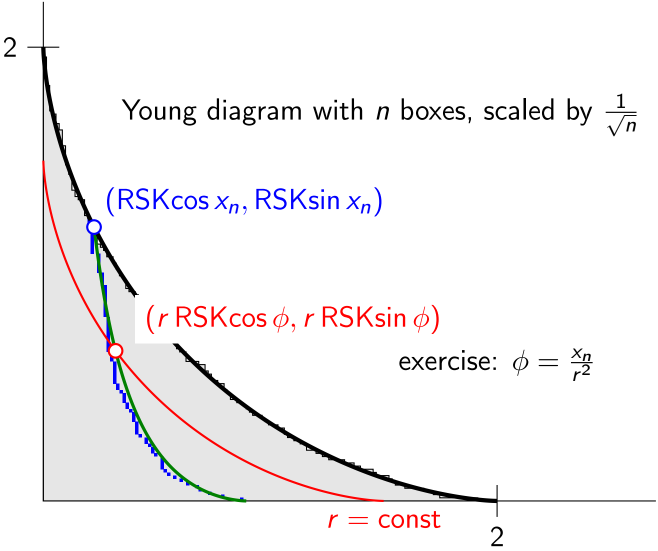
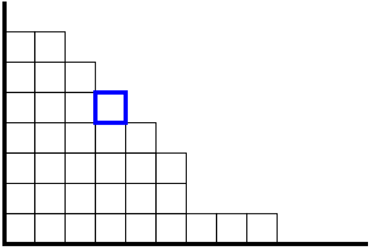


proof, part 1 — reduction of problem

instead of (for deterministic x_{n+1})

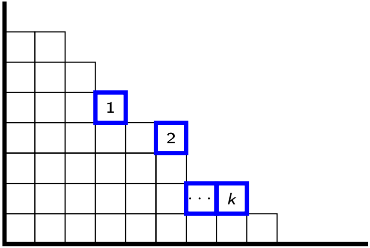
$$P(x_1, \ldots, x_n, \; x_{n+1} \qquad) \setminus P(x_1, \ldots, x_n) = \{ \; \square \; \}$$



proof, part 1 — reduction of problem

we study (for random $0 < t_1 < \cdots < t_k < 1$)

$$P(x_1, \ldots, x_n,\ t_1, \ldots, t_k) \setminus P(x_1, \ldots, x_n) = \left\{ \boxed{1}, \ldots, \boxed{k} \right\}$$



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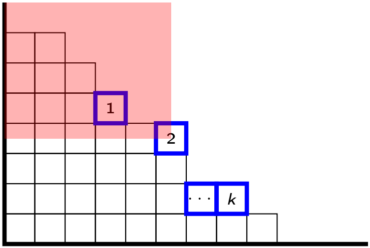


if $x_{n+1} < t_i$ then $\boxed{}$ is north-west from $\boxed{i}$

for $\frac{i}{k} \approx x_{n+1} + \epsilon$, this happens with high probability, as $k \to \infty$

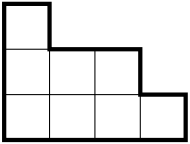
representations of the symmetric groups

representation ρ of a group G is a homomorphism to matrices

$$\rho : G \to \mathrm{GL}_k$$

---

irreducible representation ρ^λ of the symmetric group S_n $\longleftrightarrow$ Young diagram λ with n boxes



Littlewood-Richardson coefficients

$$\left(\rho^\lambda \otimes \rho^\mu\right) \Big\uparrow_{S_{|\lambda|} \times S_{|\mu|}}^{S_{|\lambda|+|\mu|}} = \bigoplus_{\nu} c^{\nu}_{\lambda,\mu}\ \rho^\nu$$

RSK and Littlewood-Richardson coefficients

if $0 \leq x_1, \dots, x_n \leq 1$ is a random sequence, such that

$$\text{shape of } P(x_1, \dots, x_n) = \lambda;$$

and $0 \leq t_1, \dots, t_k \leq 1$ is a random sequence, such that

$$\text{shape of } P(t_1, \dots, t_k) = \mu$$

then the random Young diagram

$$\text{shape of } P(x_1, \dots, x_n, t_1, \dots, t_k)$$

has the same distribution as random irreducible component of

$$V^\lambda \otimes V^\mu \Big\uparrow_{S_n \times S_k}^{S_{n+k}}$$

RSK and Littlewood-Richardson coefficients

if $0 \leq x_1, \ldots, x_n \leq 1$ is a random sequence, such that

$$\text{shape of } P(x_1, \ldots, x_n) = \lambda;$$

and $0 \leq t_1, \ldots, t_k \leq 1$ is a random sequence, such that

$$\text{shape of } P(t_1, \ldots, t_k) = (k) = \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$$

then the random Young diagram

$$\text{shape of } P(x_1, \ldots, x_n, t_1, \ldots, t_k)$$

has the same distribution as random irreducible component of

$$V^{\lambda} \otimes V^{(k)} \uparrow_{S_n \times S_k}^{S_{n+k}}$$

RSK and Littlewood-Richardson coefficients

if $0 \leq x_1, \dots, x_n \leq 1$ is a random sequence, such that

$$\text{shape of } P(x_1, \dots, x_n) = \lambda;$$

and $0 \leq t_1 < \cdots < t_k \leq 1$ is a random sequence, such that

$$\text{shape of } P(t_1, \dots, t_k) = (k) = \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$$

then the random Young diagram

$$\text{shape of } P(x_1, \dots, x_n, t_1, \dots, t_k)$$

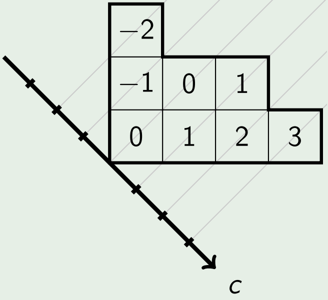
has the same distribution as random irreducible component of

$$V^\lambda \otimes V^{(k)} \big\uparrow_{S_n \times S_k}^{S_{n+k}}$$

content of the box

$$\mathrm{content}(\Box) = (x\text{-coordinate}) - (y\text{-coordinate})$$

Example



content of Young diagram $= (-2, -1, 0, 0, 1, 1, 2, 3)$

Jucys–Murphy elements

$$X_i = (1,i) + (2,i) + \cdots + (i-1,i) \qquad \text{for } i \in \{1,\dots,n\}$$

$X_1,\dots,X_n$ are elements of the symmetric group algebra $\mathbb{C}(S_n)$

for any Young diagram λ with contents $(c_1,\dots,c_n)$
and a symmetric polynomial $P(x_1,\dots,x_n)$

$$\chi^\lambda\big(P(X_1,\dots,X_n)\big) = \frac{\operatorname{Tr}\rho^\lambda\big(P(X_1,\dots,X_n)\big)}{\operatorname{Tr}\rho^\lambda(1)} = \ ?$$

Jucys–Murphy elements

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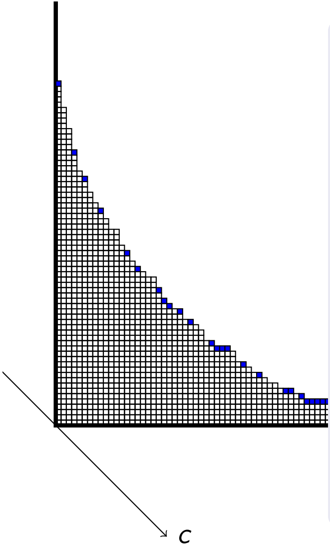
$$\chi^\lambda\big(P(X_1,\dots,X_n)\big) = \frac{\operatorname{Tr}\rho^\lambda\big(P(X_1,\dots,X_n)\big)}{\operatorname{Tr}\rho^\lambda(1)} = P(c_1,\dots,c_n)$$

growth of Young diagrams and Jucys-Murphy elements



let $\lambda \vdash n$, $\mu \vdash k$ be fixed Young diagrams

let Γ be a random irreducible component of $V^{\lambda} \otimes V^{\mu} \uparrow^{S_{n+k}}_{S_n \times S_k}$

let $c_{n+1}, \dots, c_{n+k}$ be the contents of boxes of $\Gamma \setminus \lambda$

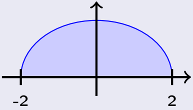
then for any symmetric polynomial $P(x_{n+1}, \dots, x_{n+k})$ we have

$$\left(\chi^{\lambda} \otimes \chi^{\mu}\right) \left(P(X_{n+1}, \dots, X_{n+k}) \downarrow^{S_{n+k}}_{S_n \times S_k}\right) = \mathbb{E} P(c_{n+1}, \dots, c_{n+k})$$

proof, part 2

if $k \approx \sqrt[4]{n}$

$$\frac{1}{k}\left(\delta_{\frac{c_1}{\sqrt{n}}} + \cdots + \delta_{\frac{c_k}{\sqrt{n}}}\right) \xrightarrow[n\to\infty]{\text{in probability}} \mu_{SC} =$$



,

where $c_i = c(\boxed{\text{i}})$

Hint: p-th moment of the left-hand-side

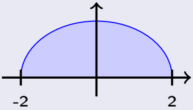
$$\frac{1}{k}\sum_j \left(\frac{c_j}{\sqrt{n}}\right)^p$$

is a random variable,
show that the mean converges to p-th moment of μ_{SC}
show that the variance converges to zero

proof, part 2

if $k \approx \sqrt[4]{n}$

$$\frac{1}{k}\left(\delta_{\frac{c_1}{\sqrt{n}}} + \cdots + \delta_{\frac{c_k}{\sqrt{n}}}\right) \xrightarrow[n\to\infty]{\text{in probability}} \mu_{SC} =$$



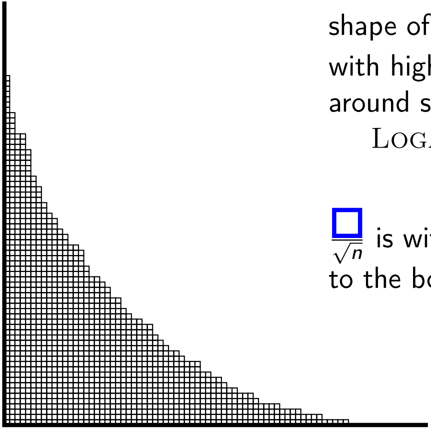
,

where $c_i = c(\boxed{\text{i}})$

since $c_1 < \cdots < c_k$, this implies that if $\frac{i}{k} \to x_{n+1}$ then

$$\frac{c(\boxed{\text{i}})}{\sqrt{n}} \xrightarrow{\text{in probability}} F^{-1}_{\mu_{SC}}(x_{n+1})$$

proof, part 3



shape of P_n (scaled by factor $\frac{1}{\sqrt{n}}$)
with high probability concentrates
around some explicit shape

Logan, Shepp, Vershik, Kerov

$\frac{\square}{\sqrt{n}}$ is with high probability close
to the boundary of this limit shape

further reading

Dan Romik, Piotr Śniady
Jeu de taquin dynamics on infinite Young tableaux and second class particles
Annals of Probability, to appear
arXiv:1111.0575

Dan Romik, Piotr Śniady
Limit shapes of bumping routes in the Robinson-Schensted correspondence
arXiv:1304.7589