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Field-theoretic approach to the Euclidean Random Assignment Problem

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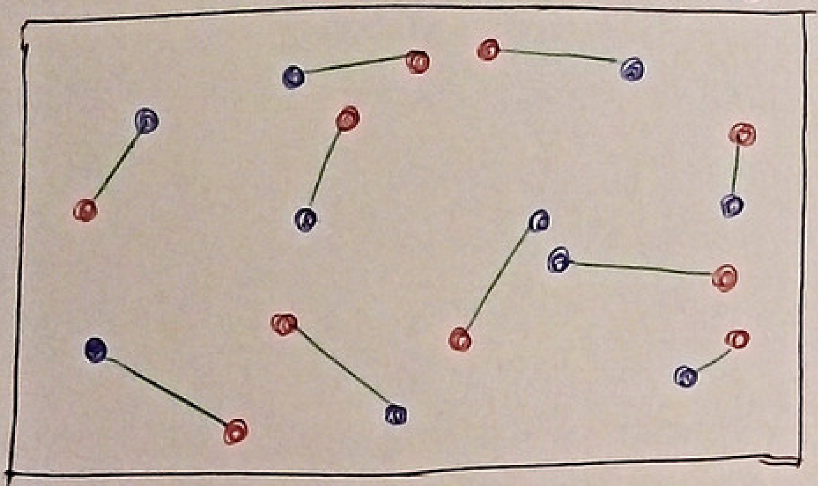


Based on various ongoing works in collaboration with

- Sergio Caracciolo (Milan Univ.)
- Matteo P. D'Achille (Univ. Paris13)
- Vittorio Erba (Milan Univ.)
- Gabriele Sicuro (ÉNS Paris)

The Assignment Problem

Let $N \in \mathbb{N}$, $C = \{C_{ij}\}$ be a $N \times N$ matrix with real-positive entries.
We search the permutation $\pi \in \mathfrak{S}_N$ which minimizes the cost function

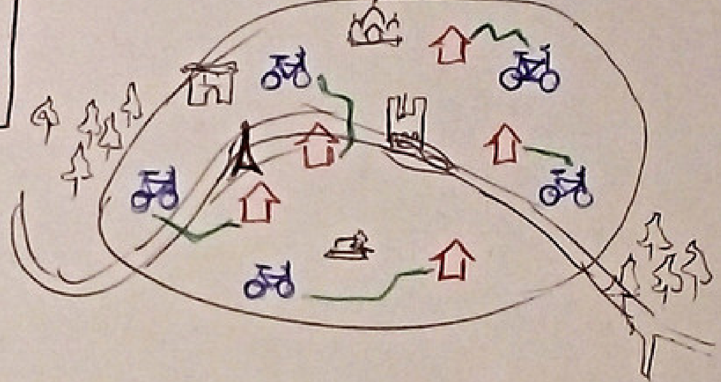
$$\boxed{\mathcal{H}_C(\pi) = \sum_{i=1}^{N} C_{i\,\pi(i)}}$$

$$
\begin{array}{c|cccccc}
 & \text{job} & & & & & \\
\text{worker} & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 9 & 3 & 2 & \boxed{0} & 7 & 9 \\
2 & 4 & 10 & 6 & 3 & 8 & \boxed{4} \\
3 & \boxed{2} & 4 & 1 & 8 & 6 & 5 \\
4 & 8 & \boxed{0} & 2 & 9 & 4 & 7 \\
5 & 5 & 4 & \boxed{3} & 6 & 7 & 6 \\
6 & 6 & 4 & 10 & 4 & \boxed{2} & 9
\end{array}
$$

Here $\boxed{\mathcal{H}_{opt}[C] := \min_{\pi} \left(\mathcal{H}_C(\pi) \right)} = 0+4+2+0+3+2 \stackrel{!}{=} 11$

$$\boxed{\{\pi_{opt}\}} := \{\pi : \mathcal{H}_C(\pi) = \mathcal{H}_{opt}[C]\} = \{(4\,6\,1\,2\,3\,5)\}$$

Notice that $\mathcal{H}_{opt}[C] > \mathcal{H}_{LB}[C] := \sum_{i=1}^{N} \min_{j}(C_{ij}) = 0+3+1+0+3+2 \stackrel{!}{=} 9$



__Example:__ when they bring back the Vélib's at night.

The Assigment Problem: history and complexity

This problem was invented, under the name of optimal transport, by Gaspard Monge in 1781.

It was "solved" algorithmically in a "lost" memory of C.G. Jacobi [1890, posthumous]
(only recognized in 2006)

Crucial contributions are:

L. Kantorovich [1942]: reformulation of Monge problem

H. Kuhn [1955]: first polynomial algorithm ($\mathcal{O}(N^4)$) "Hungarian algorithm"

Edmonds and Karp,
Tomizawa
Jonker-Volgenant

$\mathcal{O}(N^3)$ polynomial algorithms.

This problem is the "father of primal-dual algorithms" like Ford-Fulkerson, and is a paradigm of toy-model simplification of NP-complete problems:

NP-complete problem	variant which is in P	Algorithm
Travelling Salesman	Assigment Problem	Hungarian Algorithm
3-SAT	3-XOR-SAT	Gauss algorithm on GF(2)

The Random Assignment Problem

Now, let's go to Probability. Say that, for each $N \in \mathbb{N}$, you have a measure $\mu_N(C)$ on the instance $C = \{C_{ij}\}_{1 \le i,j \le N}$. This induces a measure on $\mathcal{H}_{opt}$, and various other interesting statistical quantities

$$p_N(E)\, dE = \mathbb{P}_{\mu_N}\left(\mathcal{H}_{opt}[C] \in [E, E+dE]\right)$$

$$E_N := \mathbb{E}_{p_N}(E)$$

How does it scale E_N with N, for $N \to \infty$?

First famous solution: C_{ij} i.i.d. exponential variables,

$$\boxed{E_N = \sum_{k=1}^{N} \frac{1}{k^2} \Rightarrow \lim_{N\to\infty} E_N = \frac{\pi^2}{6}}$$

Non-rigorous derivation: Mézard and Parisi, 1987

Proofs: Aldous, 1992

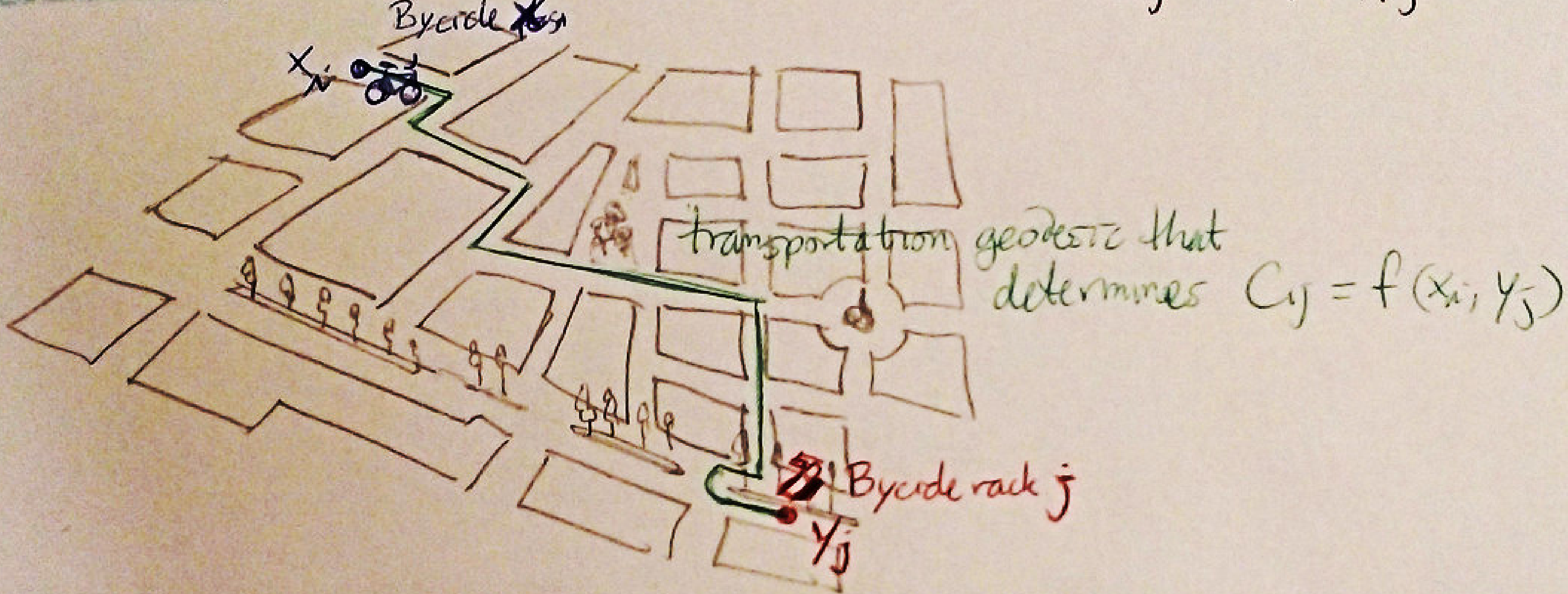
Nair, 2002 (and Nair, Prabhakar, Sharma) FOCS 2003

Linusson and Wästlund, 2003

However, when the C_{ij}'s are <u>not</u> i.i.d., the problem seems hard!

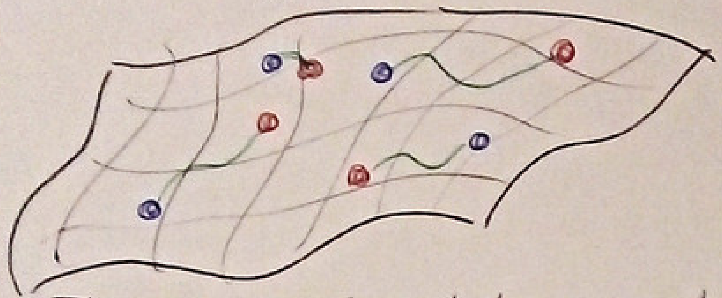
Geometric Randomness

Now, think back to the "Vélib' in Paris" optimal transport problem: the natural i.i.d variables are not the costs C_{ij} for bringing the bycicle i to the rack j, but rather the positions $x_i \in \Omega_{Paris}$ of the bycicles! Then, we have some cost function $f(x,y): \Omega_{Paris}^2 \to \mathbb{R}^+$, and, for the list of rack positions $Y = \{y_1, \ldots, y_N\}$, and byckle positions $X = \{x_1, \ldots x_N\}$, we construct the matrix of costs $C_{ij} = f(x_i, y_j)$



Geometric Randomness, local metric

The realization which is more simple to visualize, and also the most interesting, is when $X = \{x_i\}$ and $Y = \{y_j\}$ are points on a space with a metric, and $f(x_i; y_j) = f(d(x_i; y_j))$, where $d(x, y)$ is the geodesic distance



Σ manifold of real-dimension d

When $N \to \infty$, we expect that the local properties of the metric are washed out, and that the behaviour of f is dominated by the leading short-distance behaviour.

(e.g., locally the optimal matching on a sphere S_d "looks like" the one on its tangent space $\mathbb{R}^d$, and the cost function $f(d) = d^p(1 + \mathcal{O}(d^{p'}))$, $p' > 0$ is equivalent to $f'(d) = d^p$)

So, a reasonably general choice is to fix $p \in \mathbb{R}$, define $C_p(X, Y)$ as $(C_p)_{ij} = d(x_i; y_j)^p$ and $\mathcal{H}^{(p)}_{(X,Y)}(\pi) = \mathcal{H}_{C_p(X,Y)}(\pi)$

Nice fact 1: if $p=1$, $\forall (X,Y)$ $\exists$ π_{opt}, and a realization of geodesics s.t. the collection of geodesics $\gamma(x_i \to y_{\pi(i)})$ is non-crossing. ($\leftarrow$ not the case in general for all $p \neq 1$!)

Nice fact 2: $d(x,y)^p \leqslant c_p\left(d(x,z)^p + d(z,y)^p\right)$ for $c_p = \begin{cases} 1 & p \leq 1 \\ 2^{p-1} & p \geq 1 \end{cases}$

which implies $\mathcal{H}^{(p)}_{opt}(X,Y) \leqslant \left[\mathcal{H}^{(p)}_{opt}(X,Z) + \mathcal{H}^{(p)}_{opt}(Z,Y)\right] \cdot c_p$ for all Z

Nice facts at $d=1$: ($\Omega = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$) ① if $p \geqslant 1$ and $X=(x_1 \ldots x_N)$, $Y=(y_1 \ldots y_N)$ are ordered lists, then $\pi_{id} = (1,2,\ldots,N)$ is optimal. If the x_i's and y_j's are all distinct, and $p>1$, π_{opt} is unique

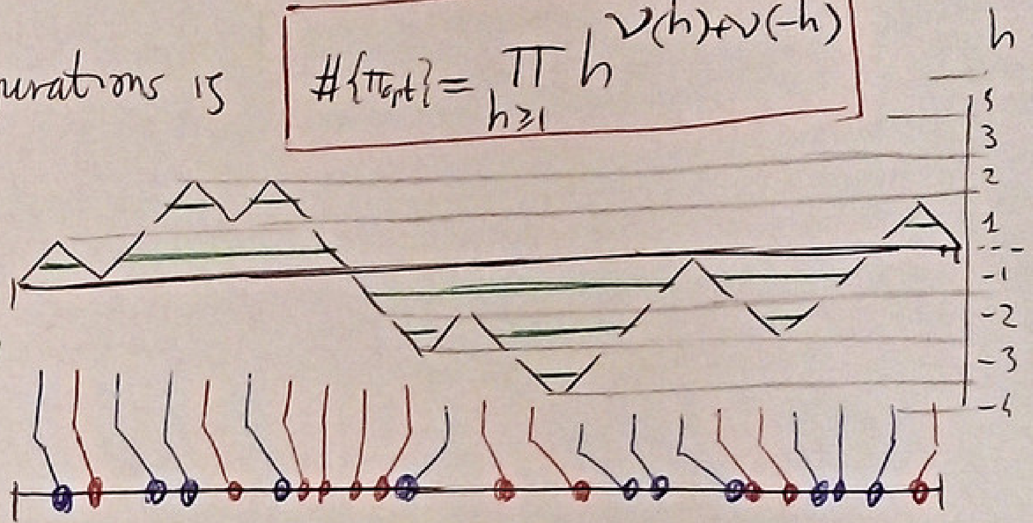
② for $p=1$, the number of optimal configurations is $\boxed{\#\{\pi_{opt}\} = \prod_{h \geq 1} h^{\nu(h)+\nu(-h)}}$

where we construct the Dyck bridge

and $\nu(h) = \#\{\text{descents at height } h\}$

③ for $p \leqslant 1$, all π_{opt} are "layered"

④ for $p = 1-\epsilon$, $\pi_{opt} = \pi_{Dyck}$



The question: understand the phase diagram!

We will concentrate on few variants of the problem:

1) for given d, Ω will be either the hypercube $[0,1]^d$, or the hypertorus $\mathbb{R}^d/\mathbb{Z}^d$.
2) $f(\text{distance}) \simeq d^p$ for $p \in \mathbb{R}$. (more precisely, $f(d) = \frac{d^p - 1}{p}$)
3) either both X and Y are i.i.d. uniform in Ω, or $N = L^d$, X is the grid with mesh spacing $1/L$, and Y entries are i.i.d. uniform (call these cases Poisson–Poisson and Grid–Poisson, respectively).

It is easily calculated that

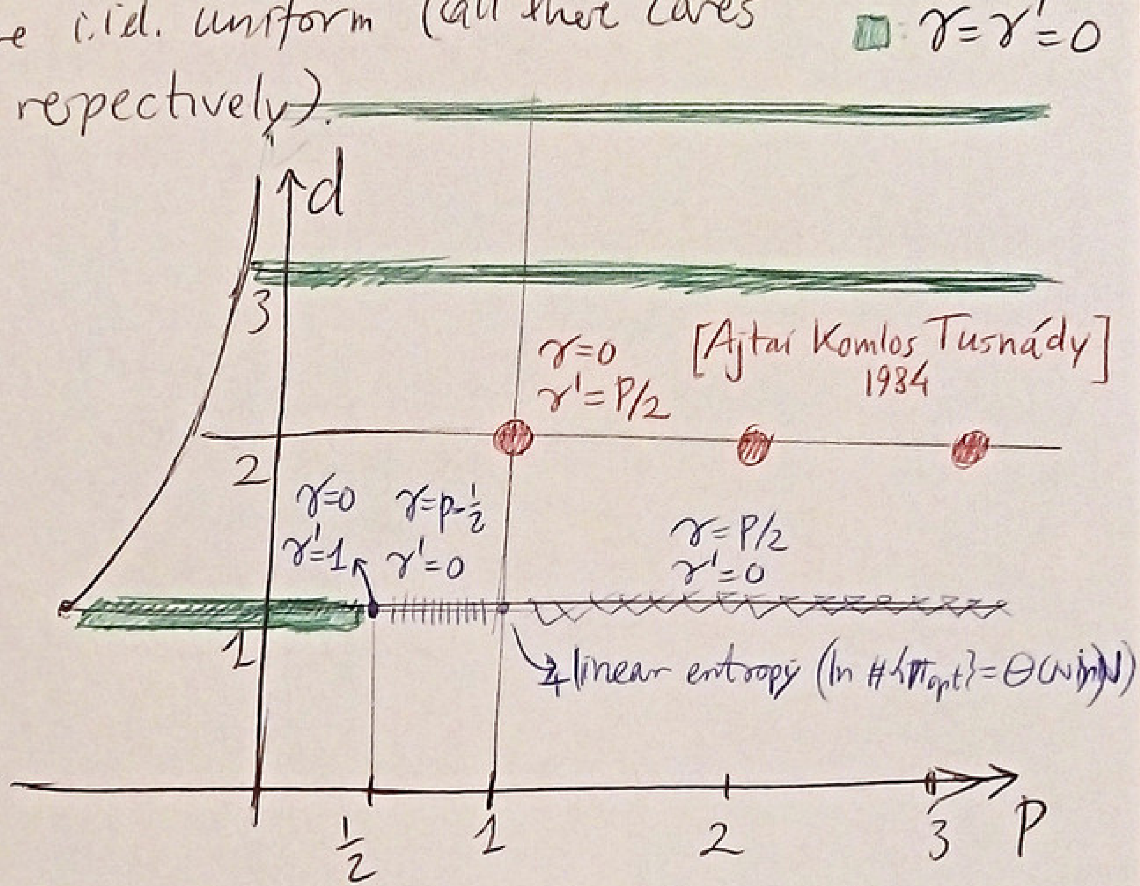
$$E_N^{(LB)} \approx N^{1 - p/d}$$

and that $\mathbb{E}(\mathcal{H}_{opt}^{(p)}) = +\infty$ if $p \leq -\frac{1}{d}$,

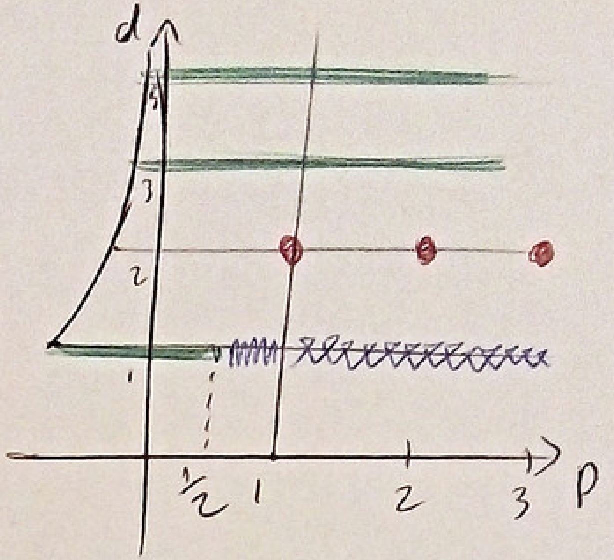
so we should try to understand

the scaling in N (of the form $C \cdot N^{\gamma} \cdot (\ln N)^{\gamma'}$?)

of $\frac{E_N}{E_N^{(LB)}}$ for $d \in \mathbb{N}$ and $p > -\frac{1}{d}$



Understanding the constants



Suppose we really know that

$$E_N / E_N^{(LB)} \sim C(d,p)\, N^{\gamma(d,p)} (\ln N)^{\gamma'(d,p)}$$

in the whole phase diagram.

Can we calculate $C(d,p)$?

YES for $d=1$, $p\geq 1$, where it is related to a certain expectation over the Brownian Bridge

NO for $\frac{1}{2}\leq p<1$ at $d=1$, but that would be interesting...

NO for the whole $\gamma=\gamma'=0$ region, and anyhow it would not be universal...

What about $d=2$? This seems (and is!) the most difficult case... However in 2014 Caracciolo, Lucibello, Parisi and Sicuro* come with a prediction, later proven by Ambrosio, Stra and Trevisan**

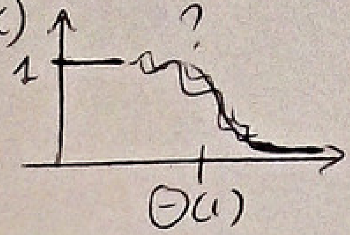
$$\boxed{C_{\text{grid-Poisson}}(d=2,p=2)=\frac{1}{2}C_{\text{Poisson-Poisson}}(d=2,p=2)=\frac{1}{4\pi}}$$

* arXiv 1402:6993, Phys Rev E 90

** arXiv 1611:04960, Prob Th Rel Fields 173

Understanding the $d=2$, $p=2$ case

In the CLPS approach, the authors, based on analogous procedures in Quantum Field Theory, postulate the existence of a cut-off function $F(x)$



such that

$$\boxed{E_N \approx \sum_{p \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(2\pi)^2} \frac{1}{|p|^2} F\left(|p|^2/N\right)}$$

Magically, despite the fact that $F(x)$ is unknown except for its sketchy behaviour ($F(x) \sim 1 + \Theta(1)$ for x small, $F(x) \approx 0$ for $x \gg \Theta(1)$, $F(x) = \Theta(1)$ for $x < \lambda$, for some $\lambda = \Theta(1)$), the leading behaviour pops up exactly!

$$E_N \approx \text{const} + \int_1^\infty dp \, \frac{1}{(2\pi)^2} \, 2\pi p \, \frac{1}{p^2} \, F\left(p^2/N\right) \approx \text{const} + \int_{1/\sqrt{N}}^\infty \frac{dp}{2\pi p} F(p^2) \approx \text{const}' + \frac{1}{4\pi} \ln N$$

However, any further term in the asymptotic expansion is out of reach by this approach...

Understanding the $d=2$, $p=2$ case

The AST approach is very complicated... It works at the level of the Monge–Kantorovich formulation, valid also for continuous distributions ρ_{Blue}, ρ_{Red} (of which our case is $\rho_{Red}(z) = \frac{1}{N}\sum_{i=1}^{N} \delta(z-x_i)$

$$\rho_{Blue}(z) = \frac{1}{N}\sum_{j=1}^{N} \delta(z-y_j)\ )$$

It crucially makes use of the triangle inequality $d(x,y)^2 \leq 2\left(d(x,z)^2 + d(z,y)^2\right)$ for separating the problem into: (*) transportation from a Dirac delta to a tight Gaussian, cost $= \Theta(1)$ ( $\sim \frac{1}{\sqrt{N}}$) (**) transportation from the Gaussian-smoothered versions of ρ_{Red} to the smoothered version of ρ_{Blue}, for which the Gaussian has introduced a cut-off factor $e^{-\alpha \cdot p^2/N}$ (beyond the $F(p^2/N)$ possibly already present). <u>But</u> (1) the proof is too hard for extracting potential terms $E_N \sim \frac{1}{4\pi}\ln N + (?) + \text{const} + \ldots$; (2) the use of triangle inequality has washed out the constant.

A stronger strategy

The problem with the CLPS approach is that it is an effective field theory arising from the linearisation of the exact action, valid for $\frac{|p|}{N} \lesssim \frac{(\ln N)^{??}}{\sqrt{N}}$

The presence of the unknown regularizator $F(p^2_N)$ is typical of this procedure, and even the "weird" logarithmic factors of AKT thm are typical of RG acting on the critical dimension, where the defining term in the action is a "marginal" operator.

In very few, very lucky cases, you can perform an exact resummation of the perturbative (Feynman diagram) series. This may be the case here, so we shall try to follow the approach of a later paper (Caracciolo-Sicuro, arXiv 1510.02320, PRL 115), but without linearization

This approach allows to describe the distribution $p_N(\mathcal{H}_{opt})$ by mean of an infinite diagrammatic series (possibly asymptotic), which is apparently much more divergent, but hopefully not after the suitable resummation...

Intermezzo: Feynman diagrams and asymptotic series

Let us approach the two subtle notions of Feynman diagrams and of asymptotic series by an illustration which is much simpler than the full-fledged QFT.

Consider $Z(g) := \int \frac{d\varphi}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\varphi^2 - \frac{g}{4!}\varphi^4}$ $\qquad F(g) = \ln Z(g)$

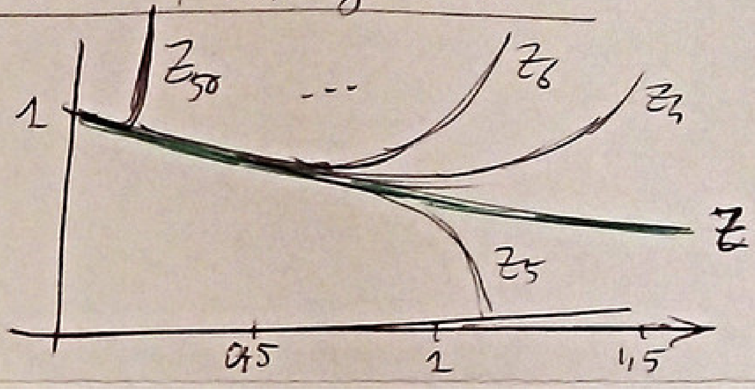
Fact: $Z(g) = +\infty$ if $g<0$, $Z(g)=1$ if $g=0$, $Z(g)$ is well-defined and monotonically decreasing for $g>0$. In fact $\boxed{Z(g) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{3}{4g}}\exp\left(\frac{3}{4g}\right) K_{\frac{1}{4}}\left(\frac{3}{4g}\right)}$ (Bessel K)

The naïve perturbative series around $g=0$ gives

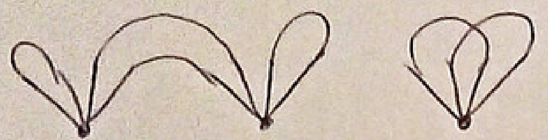
$$Z(g) = \sum_{n\geq 0} \int \frac{d\varphi}{\sqrt{2\pi}} \frac{1}{n!}\left(\frac{-g}{4!}\right)^n \varphi^{4n} e^{-\frac{\varphi^2}{2}} = \sum_{n\geq 0} \frac{(-g/4!)^n}{n!}(4n-1)!! = 1 - \frac{g}{8} + \frac{35}{384}g^2 - \frac{385}{3072}g^3 + \dots$$

$$\approx \sum_{n\geq 0} \dots \frac{1}{\pi\sqrt{2}}(n-1)!\left(-\frac{2g}{3}\right)^n$$ $\rightarrow$ This series has radius of convergence zero!

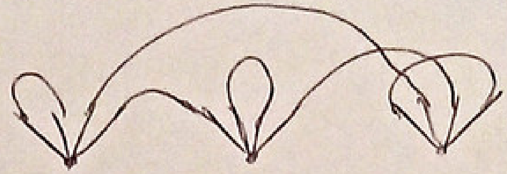
For small g, you should "cut" the pert. series at $n \sim 1/g$...



A nice fact is that the g^n term in the expansions of $F(g)$ and $Z(g)$ have a combinatorial meaning: they count (connected or general) graphs of degree 4:



typical term contributing to $Z(g)$ at order g^3



typical term contributing to $F(g)$ at order g^3

(these graphs are edge-labeled, and have a $\frac{1}{|\mathrm{Aut}|}$ symmetry factor)

The fact that the number of graphs grows more than exponentially (because of the labels) is responsible for the fact that the series is only asymptotic, and is the combinatorial counterpart of the obvious fact that

$$\int \frac{d\varphi}{\sqrt{2\pi}}\, e^{-\varphi^2/2 - \frac{g}{4!}\varphi^4} \text{ diverges if } g<0.$$

Now, you can do the same with $\varphi \to \vec{\varphi} = (\varphi_1, \dots, \varphi_n)$, $\frac{\varphi^2}{2} \to \frac{1}{2}\sum_{ij} \vec{\varphi}_i (Q^{-1})_{ij} \vec{\varphi}_j$, $g\frac{\varphi^4}{4!} \to \sum_{i_1 \dots i_4} g_{i_1 \dots i_4} \varphi_{i_1} \cdots \varphi_{i_4}$. You can diagonalise Q with a change of variables in $O(n)$, and get new coeffs $\tilde{g}_{i_1 \dots i_4}$. When Q^{-1} is "Laplacian + mass^2", this is the lattice QFT "$\lambda\varphi^4$"

A functional equation for $\mathcal{H}_{opt}(X,Y)$

Define $\boxed{Z(\beta) := \sum_{\pi\in\mathfrak{S}_N} \exp(-\beta \mathcal{H}_C(\pi)) = \operatorname{perm}\left(e^{-\beta C_{ij}}\right)}$, and $\boxed{F(\beta) = -\frac{1}{\beta}\ln Z(\beta)}$

<u>Fact</u>: for $\beta\to\infty$, $F(\beta) = \mathcal{H}_{opt} + \mathcal{O}\left(\frac{N\ln N}{\beta}\right)$

Define the <u>transport field</u> $\vec{\mu}(x_i) = \vec{Y}_{\pi(i)} - \vec{X}_i$ ✪ (if on the torus, make the obvious choice)

Then: $\rho_1(z) := \frac{1}{N}\sum_{i=1}^{N}\delta(z-x_i)$, $\rho_2(z) := \frac{1}{N}\sum_{j=1}^{N}\delta(z-y_j)$,

$\hat{\rho}_\alpha(p) := \int d^dz\, e^{i\vec{p}\cdot\vec{z}} \rho_\alpha(z)$ and π is a matching iff $\sum_i e^{i\vec{p}\cdot(\vec{x}_i+\vec{\mu}(x_i))} = \sum_j e^{i\vec{p}\cdot\vec{y}_j}$ ⊛

for all p. If X is the grid $\Lambda=\{0,\frac{1}{L},\frac{2}{L},\dots\frac{L-1}{L}\}^d$, $N=L^d$, and Ω is the torus, then the p's are the grid $\hat{\Lambda}=\{0,2\pi,2\cdot 2\pi,\dots,2\pi(L-1)\}^d$. As at $p=0$ ⊛ reduces to $N=N$, we can write

$$\sum_\pi f(\pi) = \int \mathcal{D}[\vec{\mu}]\, \overset{\text{defined by ✪}}{\tilde{f}}(\vec{\mu}) \prod_{\vec{p}\in\hat{\Lambda}\setminus\vec{0}} \delta\left(\sum_j\left(e^{i\vec{p}(\vec{x}_j+\vec{\mu}(x_j))} - e^{i\vec{p}\cdot\vec{y}_j}\right)\right)$$

Representation of Dirac deltas

We wrote $\delta\left(\sum_j \left(e^{i\vec{p}\cdot(\vec{x}_j+\vec{\mu}(\vec{x}_j))} - e^{i\vec{p}\cdot\vec{y}_j}\right)\right)$, that is $\delta(A_p(\vec{\mu}))$ for some function $A_p(\vec{\mu})$. This is a "Dirac delta", that can be seen as the limit $\epsilon \to 0$ of a tight Gaussian

$$\delta(A) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{A^2}{2\epsilon}} = \lim_{\epsilon\to 0} \int \frac{d\varphi}{2\pi} e^{-\frac{\epsilon}{2}\varphi^2 + i\varphi A}$$

Also, the "measure" $\mathcal{D}[\vec{\mu}]$ is here just the ordinary Lebesgue measure $\prod_{i=1}^{N}\prod_{\alpha=1}^{d} d(\mu_{x_i})_\alpha$. So we have, writing also $\mathcal{D}[\hat{\varphi}] = \left(\prod_{p\in\hat{\Lambda}_0} d\hat{\varphi}_p\right)\cdot(2\pi)^{-N+1}$,

$$F^{(\epsilon)}(\beta) = -\frac{1}{\beta}\ln\int \mathcal{D}[\vec{\mu}]\,\mathcal{D}[\hat{\varphi}]\; e^{-\frac{\epsilon}{2}\sum_p \hat{\varphi}_p\hat{\varphi}_{-p} + i\sum_p \hat{\varphi}_{-p}\cdot\left(\frac{1}{N}\sum_j\left(e^{i\vec{p}(\vec{x}_j+\mu(\vec{x}_j))} - e^{i\vec{p}\cdot\vec{y}_j}\right)\right)}$$

$$\cdot\, e^{-\frac{\beta}{p}\sum_i |\vec{\mu}(x_i)|^p} \qquad \left(\text{for } \mathcal{H}(\pi) = \sum \frac{|y_{\pi(i)} - x_i|^p}{p}\right)$$

When the exponent p is equal to 2 (we restrict to this case from now on), we can safely perform Fourier transform on $\vec{\mu}$, as $\sum_i |\vec{\mu}(x_i)|^2 = \sum_p \hat{\vec{\mu}}_p \cdot \hat{\vec{\mu}}_{-p}$.

The troublesome term

The term $\frac{1}{N}\sum_j e^{i\vec{p}\cdot\vec{y}_j}$ is just $\widehat{\rho_2}(p)$, where ρ_2 is the random Poisson process. If we had $\frac{1}{N}\sum_j e^{i\vec{p}\cdot\vec{x}_j}$ we would get $\widehat{\rho_1}(p) = \delta_{p,0}$ (here Kronecker delta) because ρ_1 is the deterministic grid. But we have instead

$\frac{1}{N}\sum_j e^{i\vec{p}\cdot(\vec{x}_j+\vec{\mu}(\vec{x}_j))}$. We have no other choice than Taylor expand $e^{i\vec{p}\cdot\vec{\mu}}$, hoping that $|\vec{\mu}|$ is small, but, recalling that the $\int\frac{d\varphi}{\sqrt{2\pi}}e^{-\frac{\varphi^2}{2}-\frac{g}{4!}\varphi^4}$ instructive example gave rise to an asymptotic series, we have to be careful...

$$\frac{1}{N}\sum_j e^{i\vec{p}(\vec{x}_j+\vec{\mu}(x_j))} = \frac{1}{N}\sum_{k\geq 0}\frac{i^k}{k!}\sum_{b_1..b_k\in\{1..d\}^k} p_{b_1}\cdots p_{b_k}\, e^{i\vec{p}\cdot\vec{x}_j}\,\mu_{b_1}(x_j)\cdots\mu_{b_k}(x_j)$$

$$=\sum_{k\geq 0}\frac{i^k}{k!}p_{b_1}\cdots p_{b_k}\,\mathcal{FT}[\mu_{b_1}\cdots\mu_{b_k}](\vec{p})$$

where $\mathcal{FT}(f)(p) \equiv \hat{f}(p)$ is the Fourier Transform.

This is $\sum_{k\geq 0} \frac{i^{-k}}{k!} \rho_{b_1} \cdots \rho_{b_k} \sum_{\substack{q_1 \cdots q_{k-1} \in \hat{\Lambda} \\ q_k := p-(q_1+\cdots+q_{k-1})}} \hat{\mu}_{b_1}(q_1) \cdots \hat{\mu}_{b_k}(q_k)$ (the F.T. of a product is a convolution product)

Rescale $\varphi \to \beta\varphi$. This gives

$$\boxed{\mathcal{H}_{opt}[\rho] = \lim_{\substack{\beta \gg N\ln N \\ 1/\epsilon \gg \beta}} \left\{ -\frac{1}{\beta} \ln \int \mathcal{D}[\hat{\mu}] \mathcal{D}[\hat{\varphi}] \exp\left[-\frac{\beta}{2} \epsilon\beta \sum_{p\neq 0} \hat{\varphi}(p)\hat{\varphi}(-p) + i\beta \sum_{p\neq 0} \hat{\varphi}(p)\hat{\rho}(-p) - \frac{\beta}{2} \sum_{p\in\hat{\Lambda}} \sum_{a=1}^{d} \hat{\mu}_a(p)\hat{\mu}_a(-p) - i\beta \sum_{k=0}^{\infty} \frac{i^k}{k!} \sum_{b_1 \cdots b_k} \sum_{\substack{q_1 \cdots q_k \\ p = q_1 + \cdots + q_k \\ \text{mod } \hat{\Lambda}}} \hat{\varphi}(-p) \hat{\mu}_{b_1}(q_1) \cdots \hat{\mu}_{b_k}(q_k) \rho_{b_1} \cdots \rho_{b_k} \right] + \begin{bmatrix} \text{irrelevant calculable} \\ \text{overall constant} \end{bmatrix} \right\}}$$

Call $\tilde{\epsilon} := \epsilon\beta \ll 1$

Then, the limit $\beta \to \infty$ is a "semiclassical limit", and is dominated by the stationary points of the action.

That is, calling $S[\hat{\varphi}, \hat{\mu}] = +\frac{1}{2}\tilde{\epsilon}\, \hat{\varphi}_p \hat{\varphi}_{-p} - i\, \hat{\varphi}_{-p} \hat{\rho}_p + \frac{1}{2} \hat{\vec{\mu}}_p \hat{\vec{\mu}}_{-p} + i \sum_{k\geq 0} \frac{i^k}{k!} \sum_{\substack{b_1 \cdots b_k \\ q_1 \cdots q_k}} \rho_{b_1} \cdots \rho_{b_k} \hat{\mu}(q_1) \cdots \hat{\mu}(q_k) \hat{\varphi}(-p)$

$$\mathcal{H}_{opt}[\rho] = \lim_{\substack{\beta \gg N\ln N \\ \tilde{\epsilon} \to 0}} \left\{ -\frac{1}{\beta} \ln \int \mathcal{D}[\hat{\mu}, \hat{\varphi}] \exp\left(-\beta S[\hat{\varphi}, \hat{\mu}]\right) + \text{Const}(N,\beta) \right\}$$

dominated by the solutions of $\boxed{+\frac{1}{\beta} \frac{\partial S}{\partial \hat{\varphi}(-p)} = 0, \quad +\frac{1}{\beta} \frac{\partial S}{\partial \hat{\mu}_a(-p)} = 0}$ Euler-Lagrange equations.

Our Euler-Lagrange equations

Simple manipulations on our E-L eqs, suggested by the heuristic identification of the leading terms, give

$$\boxed{\hat{\mu}_a(p) = \sum_{k\geq 0} \frac{1}{k!} \sum_{b_1\dots b_k \in \{1..d\}^k} \sum_{\substack{q_1\dots q_k \in \hat{\Lambda}^k \\ P := p - \sum q_j}} \hat{\mu}_{b_1}(q_1)\dots\hat{\mu}_{b_k}(q_k)\,\hat{\varphi}(P)\,(iP_a)(iP_{b_1})\dots(iP_{b_k}) \qquad \text{A}}$$

$$\boxed{\begin{aligned}\hat{\varphi}(p) = \frac{\hat{\rho}(p)}{|p|_0^2} &- \sum_{k\geq 1} \sum_{\substack{b_1\dots b_k \\ q_1\dots q_k \\ P = p-\sum q_j}} \frac{1}{k!} \frac{\vec{p}\cdot\vec{P}}{|p|_0^2} \hat{\mu}_{b_1}(q_1)\dots\hat{\mu}_{b_k}(q_k)\,\hat{\varphi}(P)\,(iP_{b_1})\dots(iP_{b_k}) \\ &- \frac{1}{|p|_0^2} \sum_{k\geq 2} \frac{1}{k!} \sum_{b_1\dots b_k} \sum_{\substack{q_1\dots q_{k-1} \\ q_k := p-(q_1+\dots q_{k-1})}} \hat{\mu}_{b_1}(q_1)\dots\hat{\mu}_{b_k}(q_k)\,(-iP_{b_1})\dots(-iP_{b_k}) \qquad \text{B}\end{aligned}}$$

(!) (!)

define the shortcut

$$\frac{1}{|p|_0^2} = \begin{cases} 0 & p=0 \\ \dfrac{1}{\sum_a p_a^2} & p\neq 0 \end{cases}$$

Recall that $\boxed{E_N = \mathbb{E}(\mathcal{H}_{opt}) = \mathbb{E}\left(\sum_{p\in\hat{\Lambda}} \sum_{a=1}^{d} \hat{\mu}_a(p)\hat{\mu}_a(-p)\right)}$

The average over the Disorder

Recall the definition of the cumulant generating function

$$K(t) = \ln \mathbb{E}(e^{tx}) = \sum_{n\geq 1} \frac{t^n}{n!} \kappa_n$$

$\kappa_1 = \langle x \rangle$

$\kappa_2 = \langle x^2 \rangle - \langle x \rangle^2$

$\kappa_3 = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3$

...

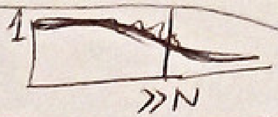
In our case

$$\ln \mathbb{E}(e^{tE_N}) = \sum_{n\geq 1} \frac{t^n}{n!} \left\langle \left(\sum_p \hat{\vec{\mu}}(p) \cdot \hat{\vec{\mu}}(-p) \right)^n \right\rangle^{\text{conn}}$$

This stands for "connected", in the sense of cumulant, which also coincides with the sense of Feynman diagrams

We have <u>one master formula</u> for $\mathbb{E}$(moments), which is in terms of the <u>$\hat{\rho}$ function</u>. Let us smear our point process to some Gaussian-like function $f(x)$ (), just like in AST, but <u>with variance $\ll \frac{1}{\sqrt{N}}$</u>, in order not to wash out our constant by triangle inequality. This turns into a cutoff as in CLPS theory, but <u>beyond the $|p|^2/N$ scale</u>, so that in order to reproduce even just the "$\frac{1}{4\pi} \ln N + ..$" result, <u>we need an exact resummation</u>

The master formula is

cutoff function 

Kronecker delta

$$\boxed{\left\langle \hat{\rho}(p_1)\cdots\hat{\rho}(p_k)\right\rangle = \prod_{j=1}^{k} F(|p_j|) \sum_{\pi\in \mathcal{J}_k^{(>1)}} \prod_{B\in\pi} N^{-|B|+1}\, \delta\Big(\sum_{j\in B}\vec{p}_j\Big)} \quad (C)$$

all p_j's $\neq 0$

set partitions of $\{1,2,\dots,k\}$ with no singletons

blocks of π

Ready to go!

The rules of the game are:

(i) write your favourite cumulant κ_c in terms of $\left\langle \sum_{p_1\dots p_c} \hat{\vec{\mu}}(p_1)\hat{\vec{\mu}}(-p_1)\dots\hat{\vec{\mu}}(p_c)\hat{\vec{\mu}}(-p_c)\right\rangle^{conn}$ (!)

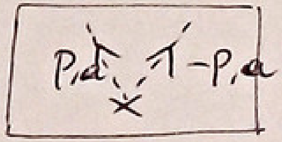
(ii) use the Euler–Lagrange equations (A) and (B) in order to rewrite
$\mu=$(series in μ,φ) , $\varphi=$(series in μ,φ,ρ), up to when only ρ's are left

(iii) use equation (C) to evaluate the average over the ρ's.

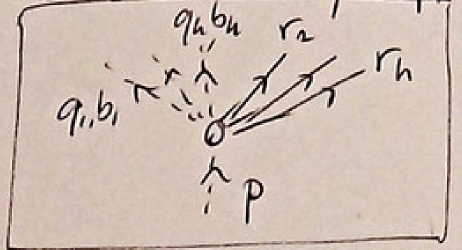
We need to transform these rules into (Feynman) graphic rules, which has the advantage of dealing with $\langle\cdots\rangle^{conn}$ automatically

The Feynman rules

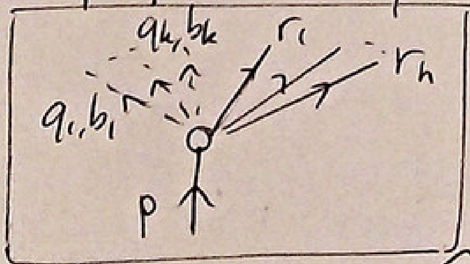
The graphic rules induced by common practice are

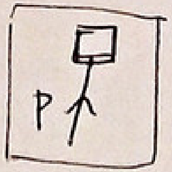
1) represent a $\hat{\vec{\mu}}(p)\,\hat{\vec{\mu}}(-p)$ factor in point (i) as a 

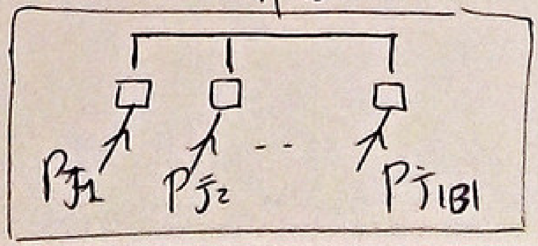
2) if you happen to have a term $\hat{\mu}_a(p) = \dots + \hat{\mu}_{b_1}(q_1)\dots\hat{\mu}_{b_k}(q_k)\,\hat{\varphi}(r_1)\dots\hat{\varphi}(r_h) + \dots$ in Ⓐ

represent this as 

3) if you have a $\hat{\varphi}(p) = \dots + \hat{\mu}_{b_1}(q_1)\dots\hat{\mu}_{b_k}(q_k)\,\hat{\varphi}(r_1)\dots\hat{\varphi}(r_h) + \dots$ in Ⓑ,

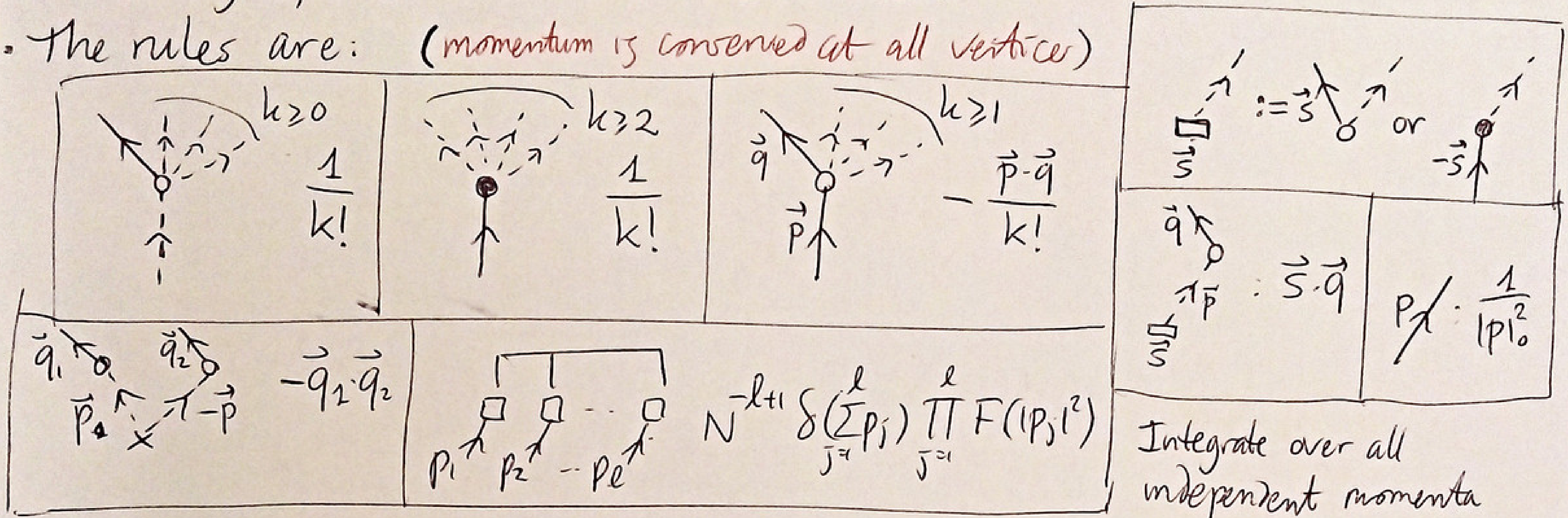
represent it as 

4) represent the only term $\hat{\varphi}(p) = \dots + \dfrac{\hat{\rho}(p)}{|p|_0^2}$ in Ⓑ as 

5) represent block factors in Ⓒ $\left(N^{-|B|+1}\,\delta\left(\sum_{j\in B} p_j\right)\right)$ as 

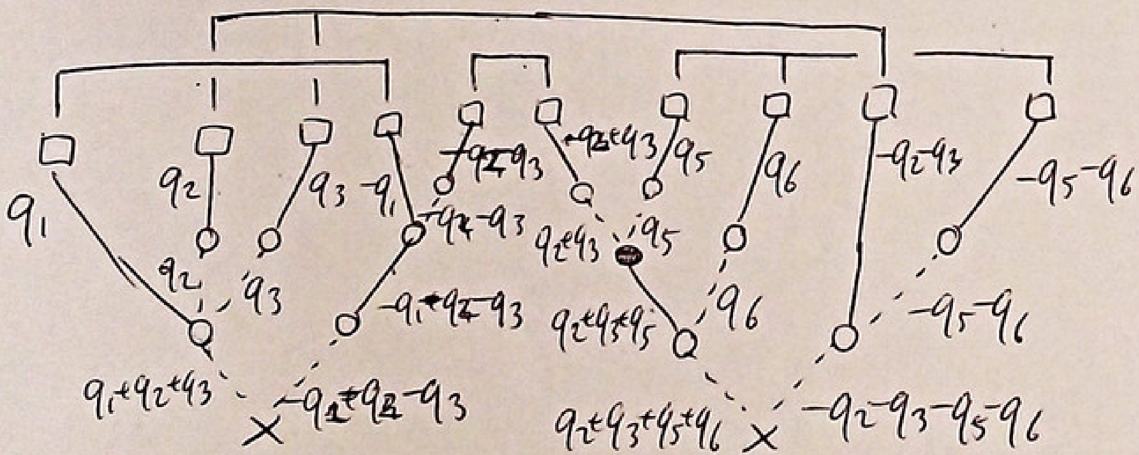
The precise rules

- Factors i "magically" combine to give quantities which are real at sight
- Indices a in $\hat{\mu}_a(p)$ "magically" combine to give only scalar products $(\vec{p}\cdot\vec{q})$ in the weights, once we sum over these indices.
- The rules are: (momentum is conserved at all vertices)



Integrate over all independent momenta

An example



$$N^{-6}\int_{q_1 q_2 \cdots q_6} (-1)^3 \cdot \frac{(q_1,q_2)(q_1,q_3)(q_1,q_2+q_3)(q_1,q_1+q_2+q_3)^2(q_2+q_3,q_2+q_3+q_5)^2(q_5,q_2+q_3+q_5)(q_6,q_2+q_3+q_5)(q_2+q_3,q_5+q_6)}{(|q_1|_0^2)^2|q_2|_0^2|q_3|_0^2(|q_2+q_3|^2)^3|q_5|^2|q_6|^2|q_5+q_6|^2\,|q_1+q_2+q_3|^2\,|q_2+q_3+q_5|^2}$$

$$\times F(q_1)^2F(q_2)F(q_3)F(q_2+q_3)^3\,F(q_5)F(q_6)F(q_5+q_6)$$

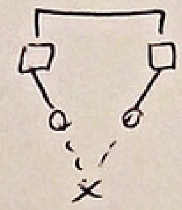
$$\times \frac{1}{(2!)^2(1!)^3}$$

...looks like a nightmare...

... but it is worse!

The bad news

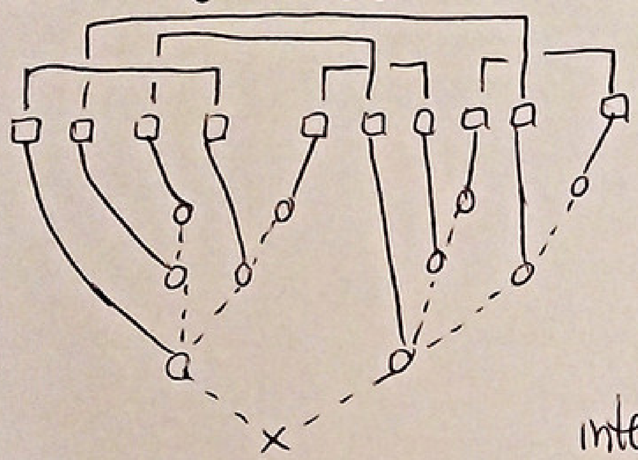
The main bad news is that, even in the simplest case $c=1$, i.e. $\mathbb{E}(E_N)$, the CLPS main singularity

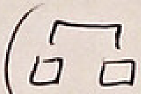
$$\frac{1}{N}\mathbb{E}(E_N) = $$  $$+ \dots = \frac{1}{N}\int_p \frac{(p,p)^2}{(|p|_0^2)^2} F(p)^2 = \frac{1}{N}\int_p \frac{1}{|p|_0^2} F(p)^2 \underset{\sim}{\geq} \frac{1}{N}\left(\frac{1}{4\pi}\ln N + \text{const}\right)$$

(from the form of the cutoff)

is not the biggest power of logarithm!

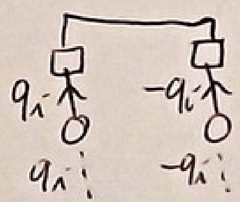
Diagrams like these



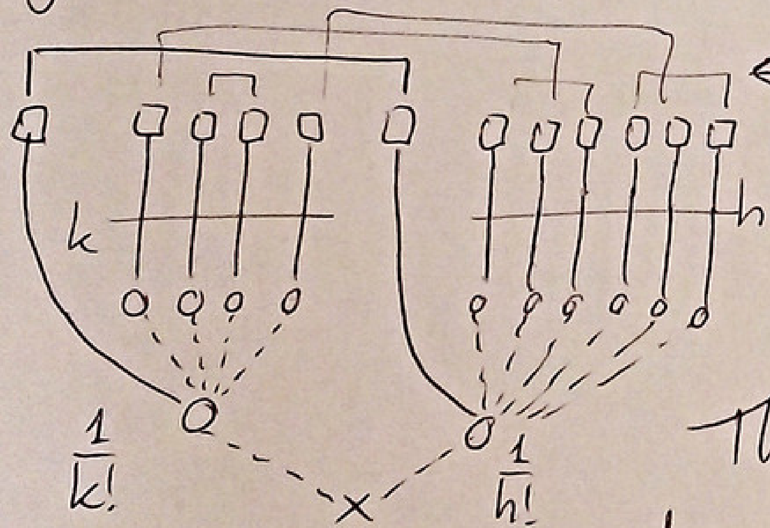
have a lot of () 2-blocks in π

any  comes with a $\frac{1}{|p|_0^2}$, and with some scalar products.

internal edges have at least 2, but edges like  may have only 1!

it follows that any pair like  comes with a diverging $\frac{1}{N}\int \frac{F(q)^2}{|q|_0^2}$

A diagram with $2n$ legs fits <u>up to $n-1$</u> such terms, like in



← any of the $(k+h-1)!!$ $\left[-(k-1)!!\,(h-1)!! \text{ if } k,h \text{ are even}\right]$ pairings which have at least one block going from left to right

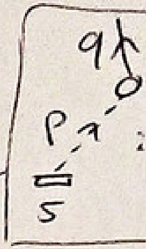
This shows that a <u>subset</u> of the Feynman diagrams contributes an (asymptotic?) series in the "small but diverging" parameter $\boxed{\frac{1}{N}\int_q \frac{F^2}{|q|^2} =: \lambda}$

(yes, $\frac{(k+h-1)!!}{k!\,h!}$ is not super-exponential, but there are plenty more dangerous terms...)

But these diagrams have signs all over... can we hope for miraculous cancellations?

<u>Encouraging fact:</u> diagrams with 3 legs have no λ factors, and diagrams with 4 legs <u>have one λ factor</u> ($\int_{\dots} = a\lambda + b$), but the coefficients combine and cancel out!

The symmetrization lemma

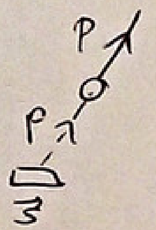
Consider one given edge $p \nearrow$ of a diagram $\mathcal{D}$. We have the rule 

: $(\vec{q} \cdot \vec{s})$

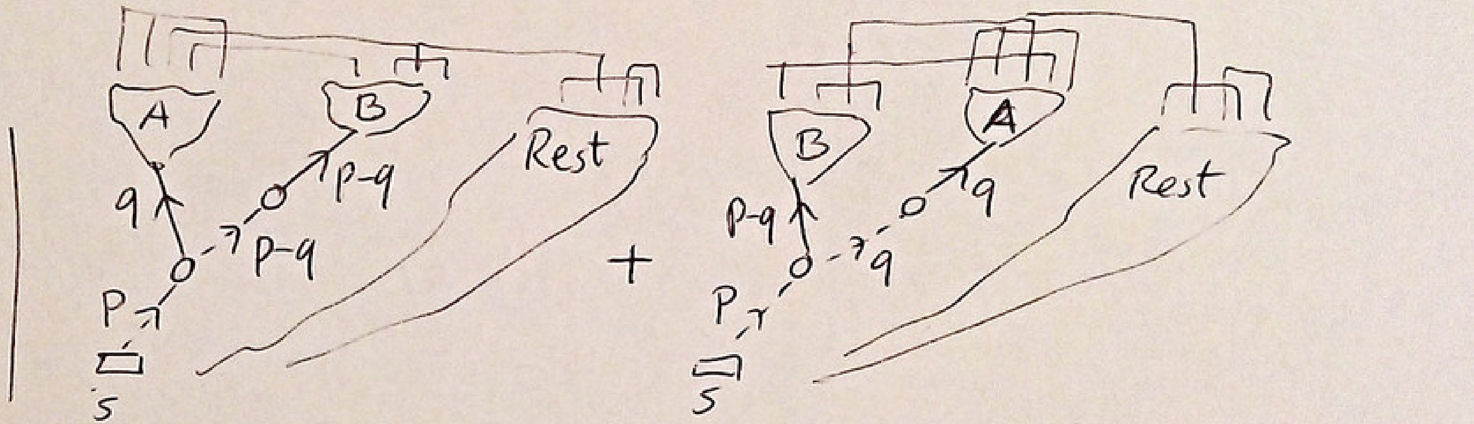
Sometimes this is annoying, we would have preferred the simpler rule : $(\vec{p} \cdot \vec{s})$.

A nice fact is that a suitable symmetrization procedure leads exactly to such a weight.

Example:



that's already $(\vec{p} \cdot \vec{s})$

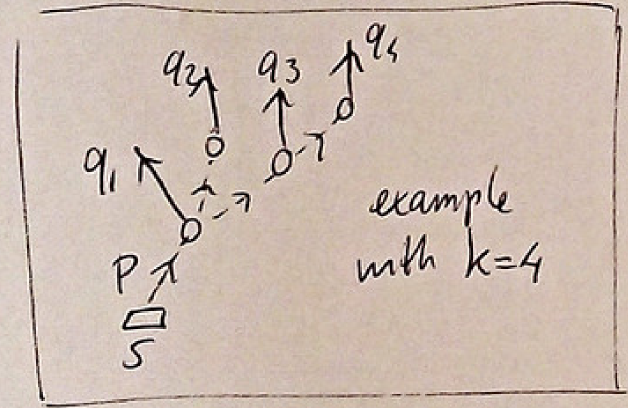


in these two diagrams <u>all other factors are the same!</u>

so we have a $\frac{1}{2!}(\vec{q}, \vec{p}-\vec{q})\left[(\vec{q}, \vec{s}) + (\vec{p}-\vec{q}, \vec{s})\right]$

$= \frac{1}{2!}(\vec{p}, \vec{s}) \cdot (\vec{q}, \vec{p}-\vec{q})$ ✓

What's the general mechanism? Say that, downstream to the edge, there are $k-1$ other dashed edges, in a tree, before all goes into solid edges...



Then we shall symmetrize over all trees, and all permutations of the outgoing momenta in solid edges. The result is

$$\boxed{\frac{1}{k!}(\vec{p}\cdot\vec{s})\cdot \mathcal{I}_k(G(\{q_j\}))}$$

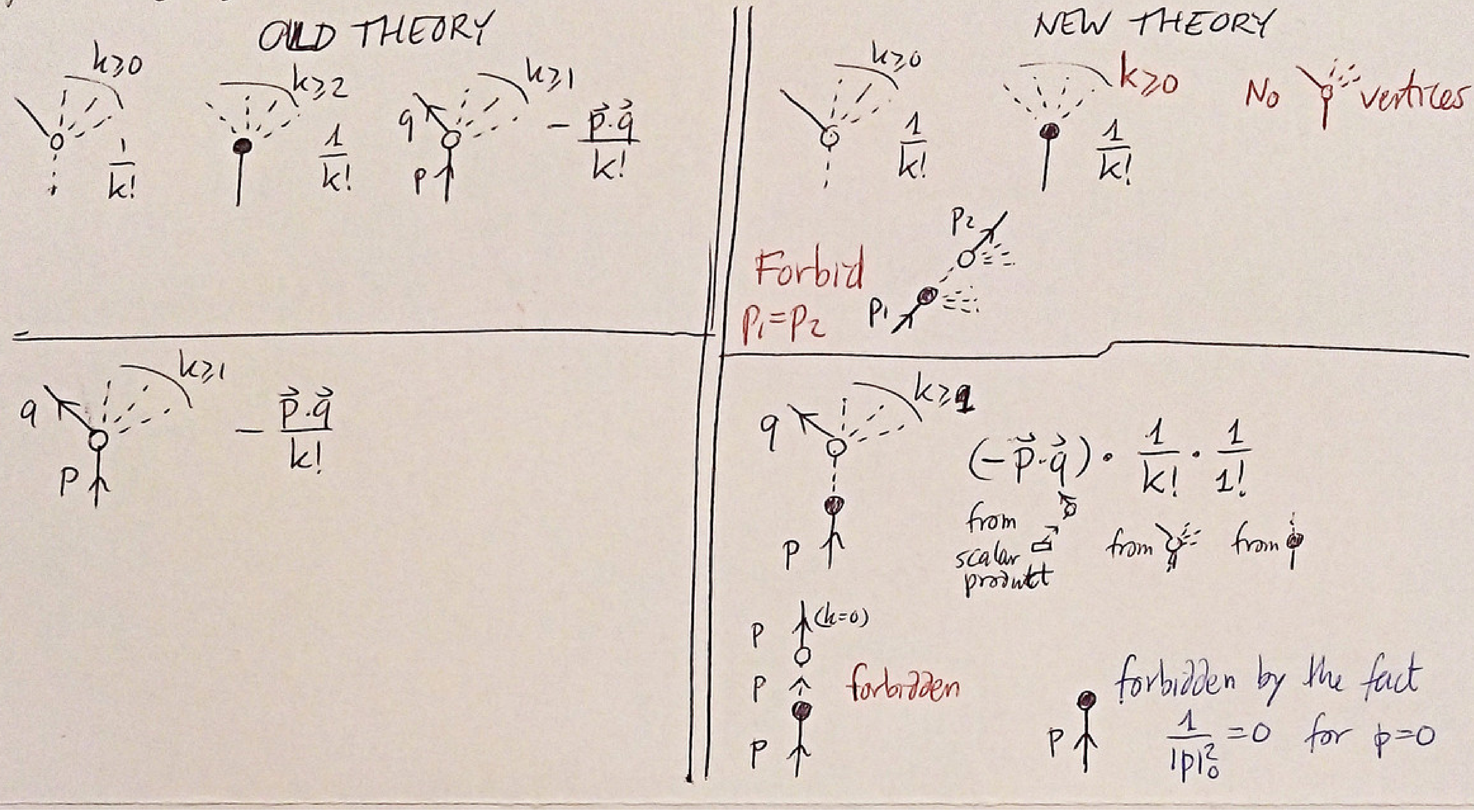
Where $G(\{q_1,\dots,q_k\})$ is the $k\times k$ Gram matrix, $G_{ij}=(\vec{q}_i\cdot\vec{q}_j)$, and $\mathcal{I}(A)$ is the spanning-tree polynomial for the set of edge-weights A_{ij}

$$L_{ij}=\begin{cases}-A_{ij} & i\neq j\\ \sum_h A_{ih} & i=j\end{cases} \qquad \mathcal{I}(A)=\det{}'L=\det L_{(1,1)} \quad \text{(by Kirchhoff Matrix-Tree Theorem)}$$

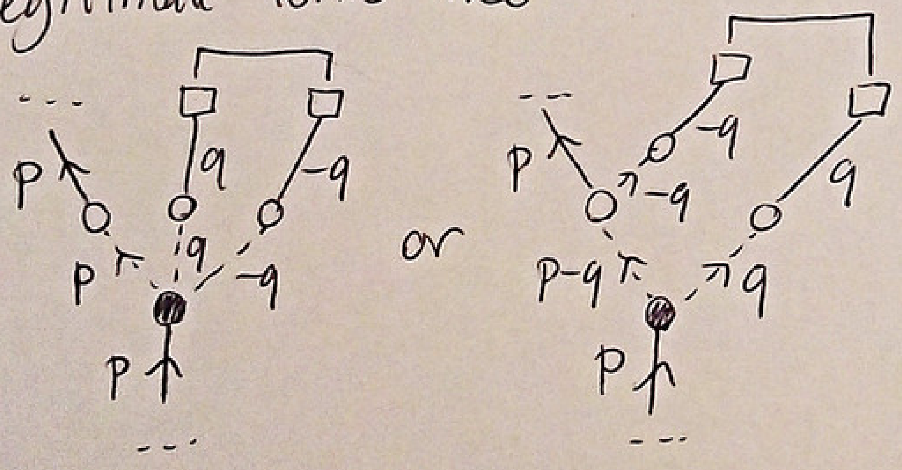
The proof is by showing that these expressions satisfy the same recursion over k.

An equivalent field theory

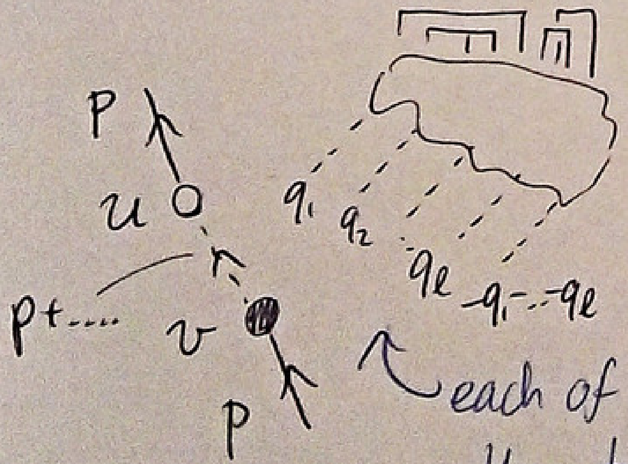
Let us now **provide** a second set of Feynman rules which, at the price of a slightly non-local rule, give the same perturbative series.



It seems that we have forbidden too much, because we have forbidden legitimate terms like



However <u>these contributions sum up to zero</u>, by symmetrizing for the possible starting point of any of the legs of the extra part:

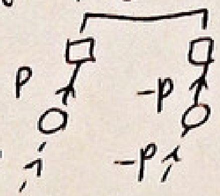


each of this legs is attached either to u or to v.

In the two cases it comes with a $(\vec{p}\cdot\vec{q}_i)$ or $(-\vec{p}\cdot\vec{q}_i)$ factor.

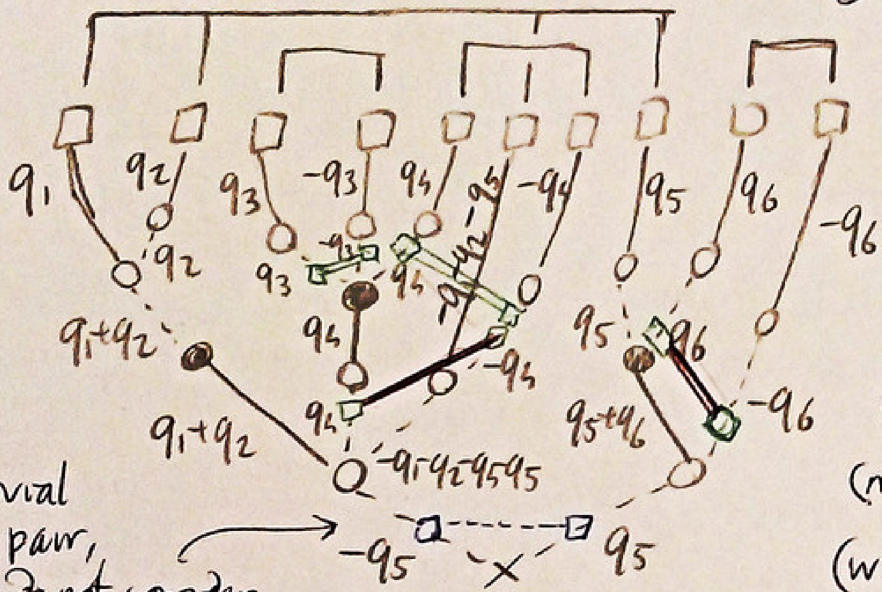
So it is legitimate to consider the perturbative series for this new theory

Resummation of log-channels

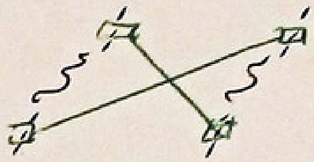
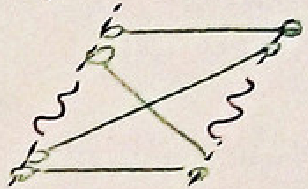
We have seen that the dangerous contributions come from portions of diagrams like  . This leads to the definition.

Two dashed edges of a diagram $\mathcal{D}$, not one ancestor of the other, are a <u>log-channel pair</u> if the set partition $\pi \in \mathcal{J}_k^{(\geq 1)}$ enforces the fact that they have opposite momenta

All the log-channel pairs of a typical large diagram. In red, the basic pairs.



the trivial log-ch. pair, that we do not consider

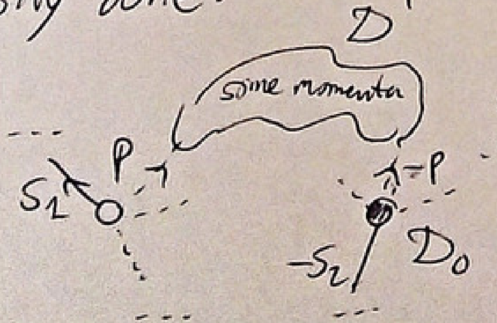
<u>Fact</u>: if  then 

A nontrivial log-channel pair (e_1, e_2) is a <u>basic pair</u> if there is no other pair (e_1', e_2') (non-trivial) with $e_1' \lneq e_1$ and $e_2' \lneq e_2$ (wrt the ordering on the tree)

Fact: any diagram $\mathcal{D}$ decomposes univocally into an irreducible diagram $\mathcal{D}_0$, and a collection of basic log-channel diagrams $\{\mathcal{D}'_j\}$.

If we prune away the log-channel diagrams, the momenta in $\mathcal{D}_0$ may not be conserved at the vertices. Call $\delta \vec{p}_v$ the momentum offset on vertex v.

Attaching/detaching a log-channel diagram $\mathcal{D}'$ to the irreducible one $\mathcal{D}_0$ is easily done:



$$I_{\mathcal{D}} = \int_{\substack{\text{momenta}\\ \text{in } \mathcal{D}_0}} f_{\mathcal{D}_0}(\dots) \int_{\substack{\text{momenta}\\ \text{in } \mathcal{D}'}} f_{\mathcal{D}'}(\dots) \cdot \underbrace{(\vec{p}\cdot\vec{s}_1)(-\vec{p}\cdot\vec{s}_2)}_{\substack{\text{because of the}\\ \text{symmetrization lemma}}}$$

Enforce the momentum offset at vertices by a representation of the delta:

$$\sum_{\substack{\mathcal{D}' \\ (v)\ \vec{p}_j \quad u\ -\vec{p}_j}} p_j = \delta p_v \quad \forall v \rightarrow \int \frac{d\vec{\xi}_v}{(2\pi)^d} e^{i\vec{\xi}_v(\delta \vec{p}_v + \sum \vec{p}_j)}$$

So we have a "gas" of log-channels (factorials just behave as they should...)

and the weight of an irred. diagram is

$$I_{\mathcal{D}_0} = \int_{\text{momenta } q_i} \int \prod_v \frac{d\vec{\xi}_v}{(2\pi)^d}\, e^{-i\sum_v \vec{\xi}_v\cdot\delta\vec{p}_v} \cdot \underbrace{e^{\frac{1}{2}\sum_{u,v}\int_p (\vec{p}\cdot\vec{s}_u)(-\vec{p}\cdot\vec{s}_v) f_{\mathcal{D}'}(\cdot)\cdot e^{i\vec{p}\cdot(\vec{\xi}_u-\vec{\xi}_v)}}}_{\circledast}$$

(inducing mom. offsets δp_v)

Call $R=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ the $\frac{\pi}{2}$ rotation. Use the fact that $p\in\hat{\Lambda} \Leftrightarrow Rp\in\hat{\Lambda} \Leftrightarrow -p\in\hat{\Lambda}$ to rewrite

$$(\vec{p}\cdot\vec{s}_1)(-\vec{p}\cdot\vec{s}_2)\, e^{i\vec{p}(\vec{\xi}_u-\vec{\xi}_v)} \longrightarrow (\vec{p}\cdot\vec{s}_1)(-\vec{p}\cdot\vec{s}_2)\cos(\vec{p}(\vec{\xi}_u-\vec{\xi}_v))$$

$$\to (\vec{p}\cdot\vec{s}_1)(-\vec{p}\cdot\vec{s}_1)\left[1-2\sin^2\left(\frac{p(\vec{\xi}_u-\vec{\xi}_v)}{2}\right)\right] = 2(\vec{p}\cdot\vec{s}_1)(\vec{p}\cdot\vec{s}_2)\sin^2\left(\frac{p(\vec{\xi}-\vec{\xi})}{2}\right)$$

$$+\frac{1}{2}\Big[\underbrace{(p\cdot s_1)(-p\cdot s_2)+(Rp\cdot s_1)(-Rp\cdot s_2)}_{=-\frac{1}{2}|p|^2 s_1\cdot s_2}\Big]$$

so that $\circledast$ becomes

$$\exp\Big[-\frac{1}{4}\underbrace{\sum_{u,v}(\vec{s}_u\cdot\vec{s}_v)}_{}\int_p |p|^2 f_{\mathcal{D}'}(\cdot) + \sum_{u,v}\int_p \overline{(\vec{p}\cdot\vec{s}_1)(\vec{p}\cdot\vec{s}_2)\sin^2\left(\frac{p(\vec{\xi}_u-\vec{\xi}_v)}{2}\right)}\, f_{\mathcal{D}'}(\cdot)\Big]$$

this term behaves as $|p|^4$ near $|p|=0$, so it is regular

This is $\left|\sum_u \vec{s}_u\right|^2$, but $\sum_u \vec{s}_u = \sum_{p_1 p_2 \cdots p_\ell} + \sum_p \;\; s_v=-p,\; s_u=p \;\; =\vec{0}$ (!)

these sum up to zero

$p-p=0$