Symmetries of triangulations

Philipp Sprüssel

TU Graz

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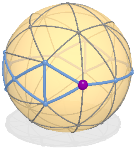
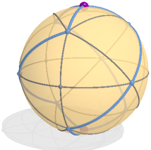
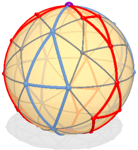
Séminaire Philippe Flajolet,
Institut Henri Poincaré

Joint work with Mihyun Kang

Labelled planar graphs

- Asymptotic number of labelled planar graphs
$$|\mathcal{P}(n)| \sim c \cdot n^{-\frac{7}{2}} \gamma^n n!, \qquad \gamma \approx 27.2$$
- Component structure of a random labelled planar graph
- Critical behaviour of a random labelled planar graph
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Question

What about unlabelled graphs?

Unlabelled planar graphs

- Asymptotic number of unlabelled planar graphs

  $$|\mathcal{P}(n)| \sim ???$$

- Component structure of a random unlabelled planar graph
- Critical behaviour of a random unlabelled planar graph
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Component structure of random graphs

$L_i(m)$ $:=$ $\#$ vertices in the *i-th largest comp.* in a random graph with n vx's and m edges, where $m = n/2 + s, \ \ s = o(n)$.

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Theorem (Bollobás 84; Łuczak 90)

- *If* $s\, n^{-2/3} \to -\infty$, *whp* $\quad L_1(m) \sim \frac{n^2}{2s^2} \log \frac{|s|^3}{n^2} = o(n^{2/3})$
- *If* $s\, n^{-2/3} \to \lambda \in (-\infty, \infty)$, *whp* $\quad L_1(m) = \Theta(n^{2/3})$
- *If* $s\, n^{-2/3} \to +\infty$, *whp* $\quad L_1(m) \sim 4\,s \gg n^{2/3}$,
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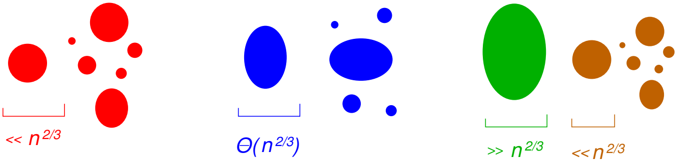

Component structure of random planar graphs

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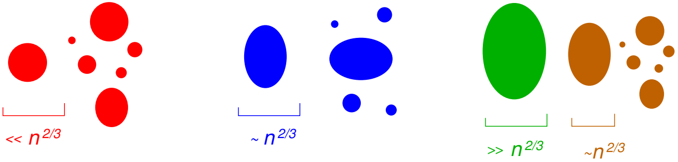

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$R(m) := \#$ vx's outside the giant component in a random planar graph with n vx's and m edges, $m = n + t, t = o(n)$.

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- *If* $t\, n^{-3/5} \to -\infty$, *whp* $\quad R(m) = (2 + o(1))|t| \gg n^{3/5}$
- *If* $t\, n^{-3/5} \to \lambda \in (-\infty, \infty)$, *whp* $\quad R(m) = \Theta(n^{3/5})$
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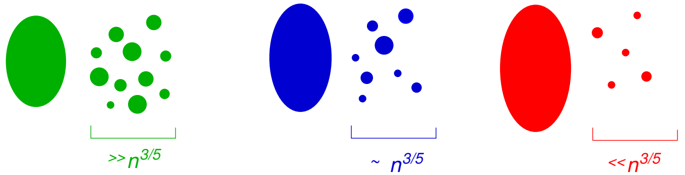

Constructions for labelled planar graphs

- Planar graphs $\longrightarrow$ Planar kernels
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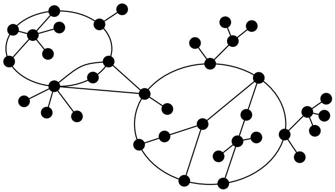

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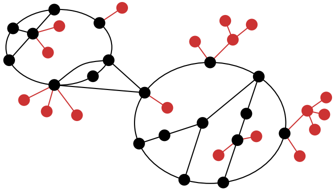

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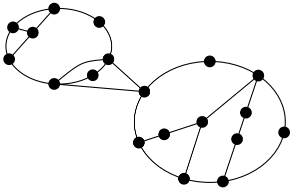

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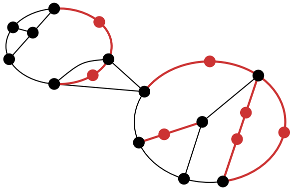

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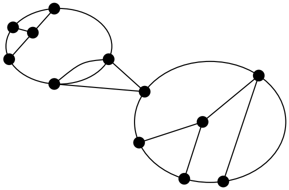

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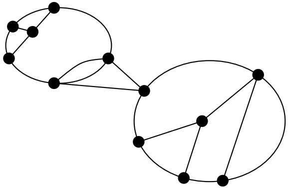

$K(x, y)$

Kernel

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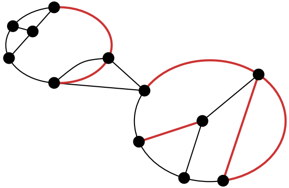

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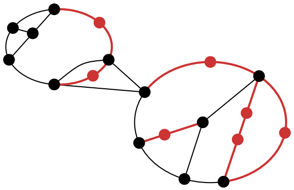

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Kernel

$C(x, y) = K(x, P(x, y))$

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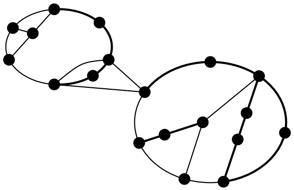

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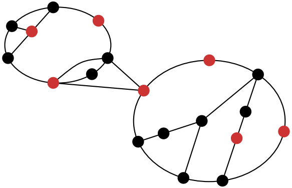

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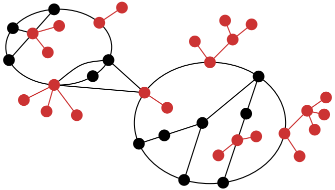

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$G(x, y) = C(T(x, y), y)$

Planar conn. graph

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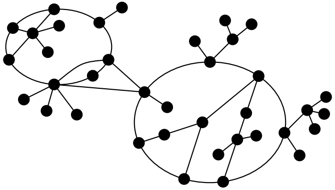

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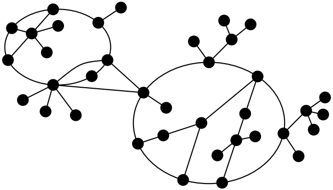

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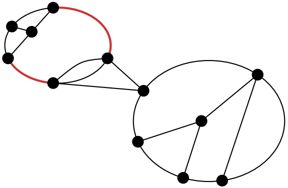
Planar conn. graph

- Planar graphs $\overset{\text{Whitney}}{\longleftrightarrow}$ 3-conn. cubic planar graphs $\overset{\text{Dual}}{\longleftrightarrow}$ 3-conn. cubic maps $\longleftrightarrow$ Simple plane triangulations

Unlabelled planar graphs

Problem: Indistinguishable vertices/edges



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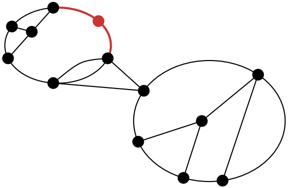

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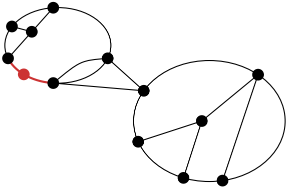

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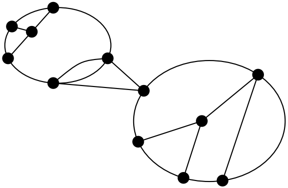

Solution: Cycle index sums

Unlabelled planar graphs

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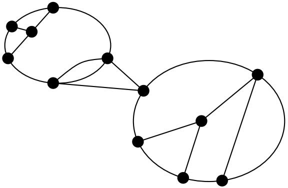

Solution: Cycle index sums

Building blocks $x_1^{a_1} x_2^{a_2} \cdots y_1^{b_1} y_2^{b_2} \cdots$

Information about sizes of orbits $\forall f \in \mathrm{Aut}(G)$

Unlabelled planar graphs

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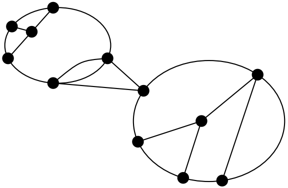

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Replacements similar to GFs

Unlabelled planar graphs

With cycle index sums:

Unlabelled planar graphs $\longleftrightarrow$ $\cdots$ $\longleftrightarrow$ Triangulations

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But: different factors depending on symmetries.

Unlabelled planar graphs

With cycle index sums:

$$\text{Unlabelled planar graphs} \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \text{Triangulations}$$

But: different factors depending on symmetries.

Problem

Describe the triangulations with a given set of symmetries.

Unlabelled Triangulations

Notation

- *Cells of dim 0,1,2: vertices, edges, and faces*
- $\mathrm{Aut}(c, T)$: *all automorphisms of T that fix a given cell c*

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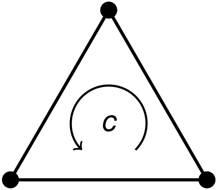
Properties of automorphisms

- $\varphi \in \mathrm{Aut}(c, T)$: uniquely determined by its action on the cells incident with c
- $\mathrm{Aut}(c, T)$ is isomorphic to a subgroup of the dihedral group $D_{\deg(c)}$

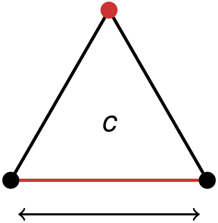
Unlabelled triangulations

Two types of non-trivial automorphisms:

Rotations
(no invariant cells adj. to c)



Reflections
(two invariant cells opp. at c)



Unlabelled triangulations

Symmetries of triangulations (Kang & Sprüssel 15+)

- If $\mathrm{Aut}(c, T)$ contains a reflection but no rotation, then it is isomorphic to the 2-element group $\mathbb{Z}_2$.
- If $\mathrm{Aut}(c, T)$ contains $k \geq 1$ rotations but no reflection, then it is isomorphic to the cyclic group $\mathbb{Z}_{k+1}$.
- If $\mathrm{Aut}(c, T)$ contains both reflections and rotations, then it is isomorphic to a dihedral group D_m with $m \mid \deg(c)$.

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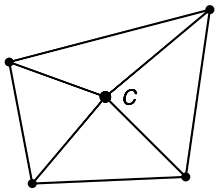
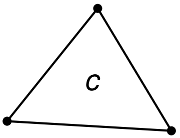
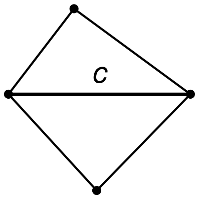


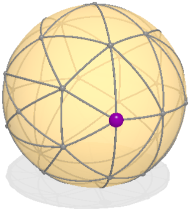




Triangulations with reflective symmetries

Theorem (Tutte 62)

The invariant cells of a reflection are the elements of a cyclic sequence $C = (c_1, \ldots, c_\ell)$ s.t. for each cell c_i, its predecessor and its successor in C lie opposite at c_i.



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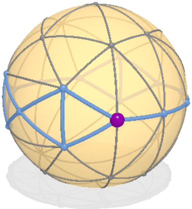

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Girdle G: all vx's & edges in C and on the b'daries of faces in C

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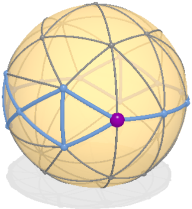

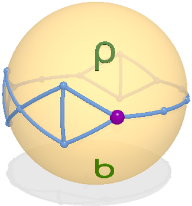

Definition

Girdle G: all vx's & edges in C and on the b'daries of faces in C

$\Longrightarrow$ induces two near-triangulations ρ

Triangulations with reflective symmetries

Theorem (K-S 15+)

The triangulations with a reflective but no rotative symmetry are precisely the ones obtained by choosing

- *a girdle G and*
- *a near-triangulation ρ with forbidden chords*

and attaching a copy of ρ into both sides of G. This is a 2-to-1 correspondence.

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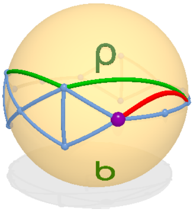
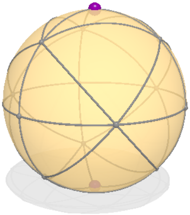


Lemma (Tutte 62)

Every rotative automorphism φ has precisely one invariant cell $c' \neq c$.



Triangulations with rotative symmetries

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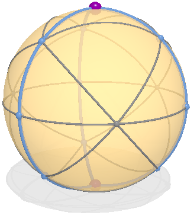

Definition

Spindle S: union of paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ (m order of φ)

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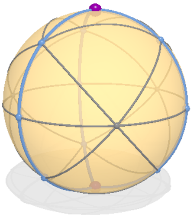
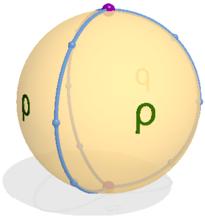




Definition

Spindle S: union of paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ (m order of φ)

$\Longrightarrow$ induces m isomorphic near-triangulations ρ

Theorem (K-S 15+)

The triangulations with a rotative symmetry are precisely the ones obtained by choosing

- *a spindle S and*
- *a near-triangulation* ρ

and attaching a copy of ρ *into each segment of S.*

Triangulations with rotative symmetries

Theorem (K-S 15+)

The triangulations with a rotative symmetry are precisely the ones obtained by choosing

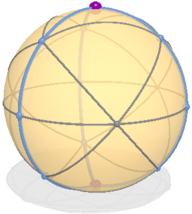
- *a spindle S and*
- *a near-triangulation ρ*

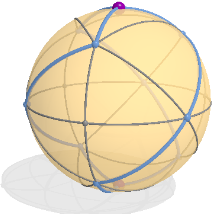
and attaching a copy of ρ into each segment of S.

But: Every triangulation corresponds to a different number of spindles and near-triangulations.

Triangulations with rotative symmetries

Different spindles & near-triangulations for the same triangulation:





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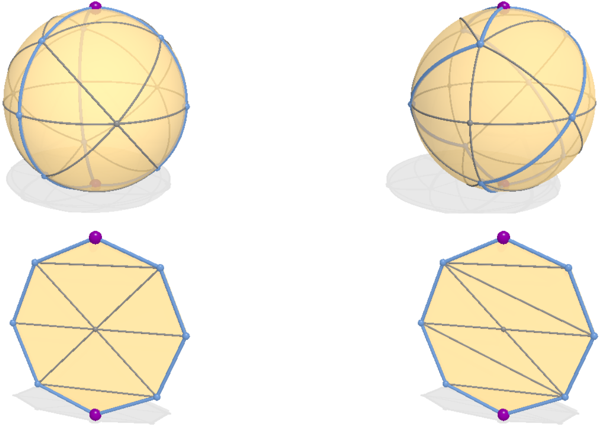
Different spindles & near-triangulations for the same triangulation:



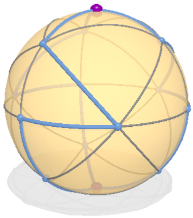
Idea: Eliminate the element of choice in the construction of the spindle.

Triangulations with rotative symmetries

Idea: Eliminate the element of choice in the construction of the spindle.

Construct spindle S from north to south:

- Take all edges going out of c;
- Take the leftmost edge for each path;
- Go right as far as possible;
- Iterate until you reach c'.



Triangulations with rotative symmetries

Definition (K-S 15+)

Extended spindle S: Defined iteratively from north to south.

Triangulations with rotative symmetries

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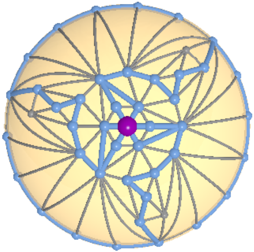
Extended spindle might have "bubbles".



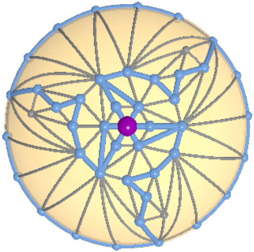
Triangulations with rotative symmetries

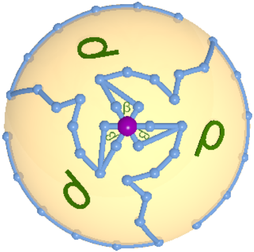
Definition (K-S 15+)

Extended spindle S: Defined iteratively from north to south.

Extended spindle might have "bubbles".

$\Longrightarrow$ induces sets of m isomorphic near-triangulations $\rho, \beta, \ldots$





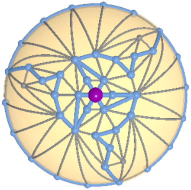
Triangulations with rotative symmetries

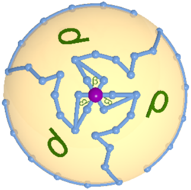
Theorem (K-S 15+)

The triangulations with a rotative symmetry are precisely the ones obtained by choosing

- *an extended spindle S,*
- *a near-triangulation ρ with additional structure, and*
- *near-triangulations $\beta_1, \ldots, \beta_\ell$*

and attaching copies of ρ into each segment of S and copies of $\beta_1, \ldots, \beta_\ell$ into each bubble of S. This is a 1-1 correspondence.





Reflective and rotative symmetries

Reminder

- If there are both reflections and rotations, then $\mathrm{Aut}(c, T)$ is isomorphic to a dihedral group D_k with $k \mid \deg(c)$.
- D_k contains k reflections and $k - 1$ rotations.
- Every reflection has a girdle.
- For every rotation $\exists$ a unique invariant cell $c' \neq c$.

Reflective and rotative symmetries

Reminder

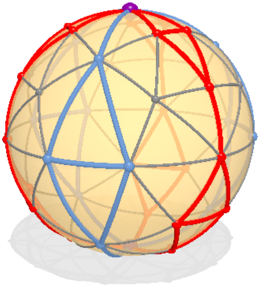
- If there are both reflections and rotations, then $\mathrm{Aut}(c, T)$ is isomorphic to a dihedral group D_k with $k \mid \deg(c)$.
- D_k contains k reflections and $k - 1$ rotations.
- Every reflection has a girdle.
- For every rotation $\exists$ a unique invariant cell $c' \neq c$.

- c' is the same for all rotations.
- Girdles intersect only in c and c'.
- Every second girdle is isomorphic.

Reflective and rotative symmetries

Definition (K-S 15+)

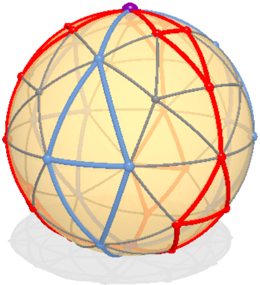
Skeleton S: union of the k girdles

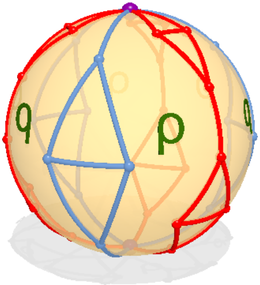


Definition (K-S 15+)

Skeleton S: union of the k girdles

$\Longrightarrow$ induces isomorphic near-triangulations ρ



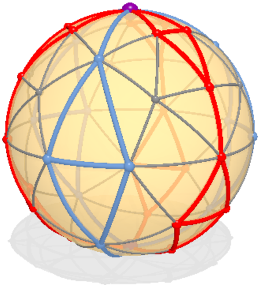


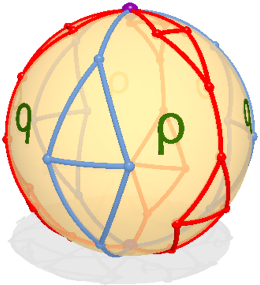
Reflective and rotative symmetries

Definition (K-S 15+)

Skeleton S: union of the k girdles

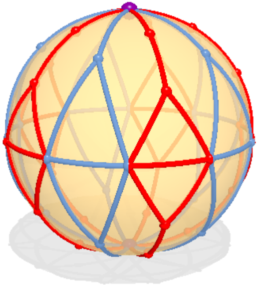
$\Longrightarrow$ induces isomorphic ~~near-triangulations ρ~~

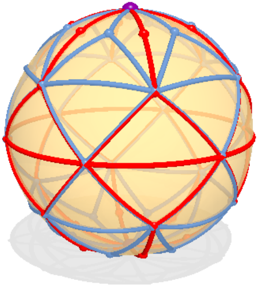




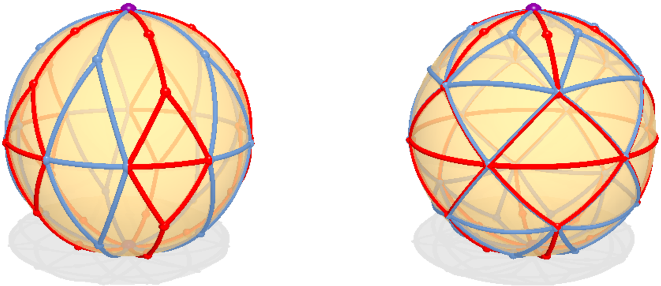
Reflective and rotative symmetries

Girdles can touch:





Girdles can touch:



~~isomorphic near-triangulations ρ~~
near-triangulations $\rho_1, \dots, \rho_\ell$, each appearing $2k$ times

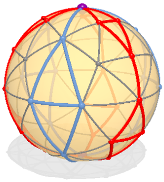
Reflective and rotative symmetries

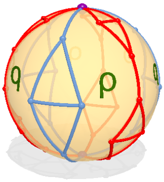
Theorem (K-S 15+)

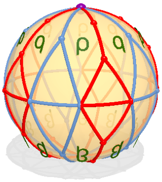
The triangulations with reflective and rotative symmetry are precisely the ones obtained by choosing

- *a skeleton S and*
- *near-triangulations* $\rho_1, \ldots, \rho_\ell$ *with forbidden chords*

and attaching copies of $\rho_1, \ldots, \rho_\ell$ *into each segment of S. This is a 2-1 correspondence.*



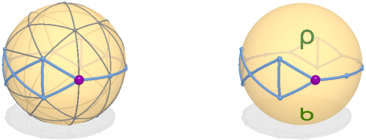




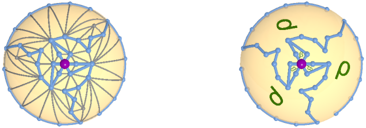
Summary

Characterization of symmetries of triangulations

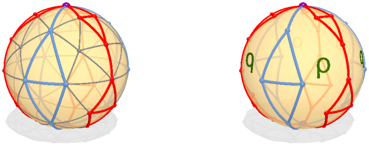
- Reflective:
  Girdle



- Rotative:
  (Extended) spindle



- Reflective & rotative:
  Skeleton



Summary and Outlook

Details:

- Cycle index sums for girdles, spindles, and skeletons;
- Decomposition scheme for near-triangulations;
- Cycle index sums for near-triangulations.

Summary and Outlook

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- Cycle index sums for girdles, spindles, and skeletons;
- Decomposition scheme for near-triangulations;
- Cycle index sums for near-triangulations.

Outlook:

- Transfer cycle index sums to cubic 3-conn. maps
- 3-conn. cubic maps $\longrightarrow$ 3-conn. cubic planar graphs

  $\dots$

  $\longrightarrow$ Unlabelled planar graphs
- Asymptotic numbers

The end