PROOF OF THE ASM-DPP CONJECTURE

(PDF + Roger Behrend + Paul Zinn-Justin)

- Physical Combinatorics

PROOF OF THE ASM-DPP CONJECTURE

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- Physical Combinatorics
- Descending Plane Partitions **DPP**

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- Physical Combinatorics
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- Alternating Sign Matrices

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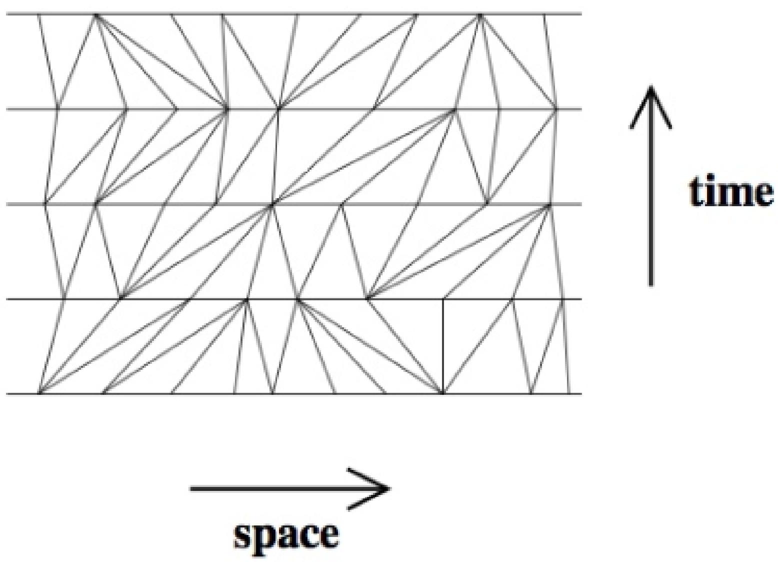
identity between refined enumerations

det(DPP) = det(ASM)

DIGRESSION: 1+1 Dimensional Lorentzian quantum gravity

(PDF + Emmanuel Guitter + Charlotte Kristjansen '99)



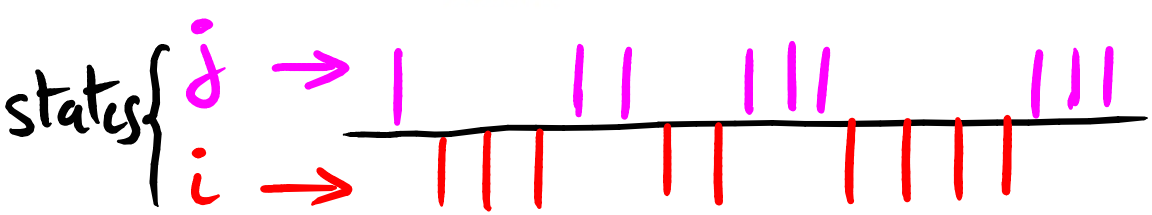
Triangulations that are

1. Random in space direction
2. Regular in time direction

$\Rightarrow$ **TRANSFER MATRIX**

$$T_{ij} = \binom{i+j}{i}$$

$(i, j \in \mathbb{Z}_+)$



Include { curvature weight a / $\perp\!\perp$ or $\top\!\top$
{ area weight g / $\perp$ or $\top$

Then

$$T_{i,j}(g,a) = (ag)^{i+j} \sum_{k=0}^{\mathrm{Min}(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k}$$

Generating Function

$$\sum_{i,j\geq 0} z^i w^j T_{i,j}(g,a) = \frac{1}{1-ga(z+w)-g^2(1-a^2)zw}$$

Integrability

$$\left[T(g,a)\,,\,T(g',a')\right] = 0$$

$$\Longleftrightarrow \quad \varphi(g,a) = \varphi(g',a')$$

$$\varphi(g,a) = \frac{1-g^2(1-a^2)}{ag} \qquad (= q+q^{-1})$$

END OF DIGRESSION

DPP = Arrays of positive integers

$$\begin{array}{ccccccc} a_{11} & a_{12} & \cdots & & & & a_{1,\mu_1} \\ & a_{22} & a_{23} & \cdots & & a_{2,\mu_2} & \\ & & \ddots & & & & \\ & & & a_{r,r} & \cdots & a_{r,\mu_r} & \end{array}$$

1. $a \geq b$ (→)
2. $a > b$ (↓)
3. $\lambda_i = \mu_i + i - 1$ = # parts in row i

   $\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $a_{ij} \leq j-i$ = special part
- order n = $a_{ij} \leq n \ \forall i,j$.

OBSERVABLES

parts = n
special parts
non-special parts

$n=3$ — 7 DPP's

$\emptyset$ [2]   [3]   [3 3]   [3 2]   [3 1]   [3 3 / 2]

DPP = Arrays of positive integers

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Vocabulary

- a_{ij} = part
- $a_{ij} \leq j - i$ = special part
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OBSERVABLES

parts = n → (3)

special parts →

non-special parts

(n=3) 7 DPP's

∅ | 2 | 3 | 3 3 | 3 2 | 3 | 3 3 / 2

ASM

- $n \times n$ matrices with elements $0, \pm 1$
- ± 1 alternate along rows and columns
- row and column sums $= 1$

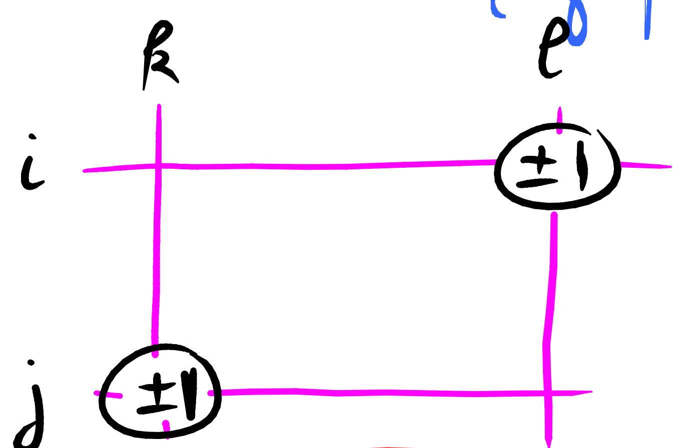
Generalize Permutation matrices (λ-determinant of Mills-Robbins-Rumsey)

$n=3$: 7 ASM's

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

OBSERVABLES

- position of the 1 in the first row
- #(−1) number of −1's
- inversion number (of permutations)



$$\mathrm{inv}(A) = \sum_{i<j} \sum_{k<\ell} A_{i\ell} A_{jk}$$

THE ASM–DPP CONJECTURE

JOURNAL OF COMBINATORIAL THEORY, Series A 34, 340–359 (1983)

Alternating Sign Matrices and Descending Plane Partitions

W. H. Mills, David P. Robbins, and Howard Rumsey, Jr.

Institute for Defense Analyses, Princeton, New Jersey 08540-3699

Communicated by the Managing Editors

Received March 15, 1982

Conjecture 3. Suppose that n, k, m, p are nonnegative integers, $1 \leqslant k \leqslant n$. Let $\mathscr{A}(n, k, m, p)$ be the set of alternating sign matrices such that

- (i) the size of the matrix is $n \times n$,
- (ii) the 1 in the top row occurs in position k,
- (iii) the number of -1's in the matrix is m,
- (iv) the number of inversions in the matrix is p.

On the other hand, let $\mathscr{D}(n, k, m, p)$ be the set of descending plane partitions such that

- (I) no parts exceed n,
- (II) there are exactly $k-1$ parts equal to n,
- (III) there are exactly m special parts,
- (IV) there are a total of p parts.

Then $\mathscr{A}(n, k, m, p)$ and $\mathscr{D}(n, k, m, p)$ have the same cardinality.

ASM (A(n))
- n = size
- position top 1
- # −1's
- # inversions

DPP (D(n))
- n = order
- # parts = n
- # special parts
- # parts

Counting

- DPP : Andrews '79 → formula $D(n)$
- ASM : Zeilberger '96, Kuperberg '96 → $A(n) = \mathrm{TSSCPP}(n) = D(n)$

6 vertex model

↑

Izergin–Korepin integrable lattice model

Refinements?

Computation of the refined numbers

Strategy. 1. reformulate in terms of known (and manageable objects)

- DPP → lattice paths (lattice fermions)
- ASM → 6 vertex model (integrable lattice model)

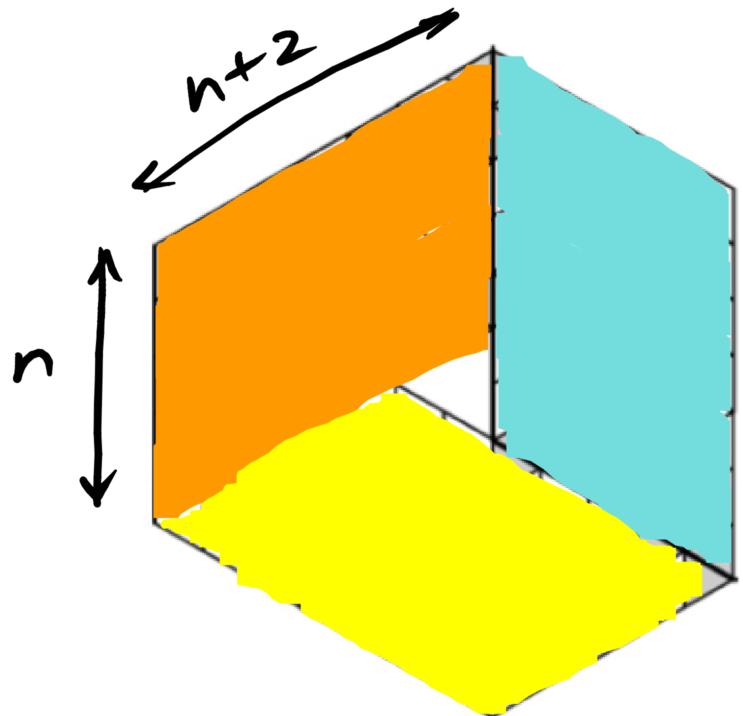
2. write refined generating functions as determinants

3. Prove identity between determinants

DPP

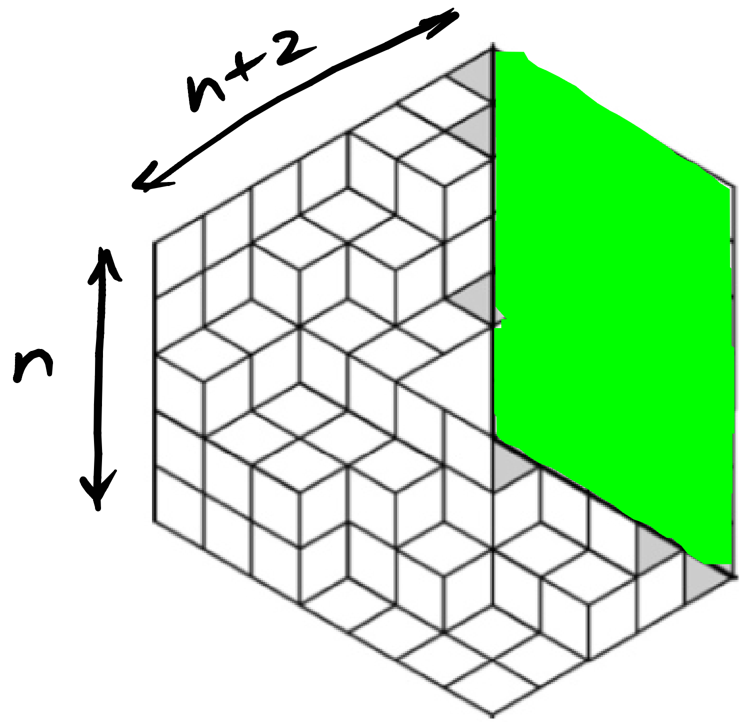
From DPP to Lattice Paths

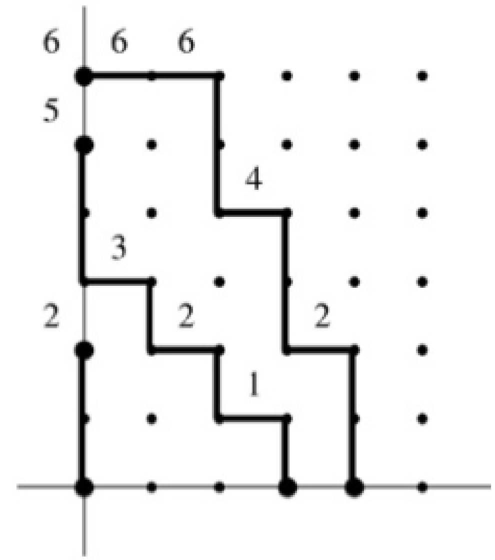
Lalonde '03
(Krattenthaler '06)



Cyclically symmetric Rhombus tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole $\Leftrightarrow$ Lattice Paths

From DPP to Lattice Paths

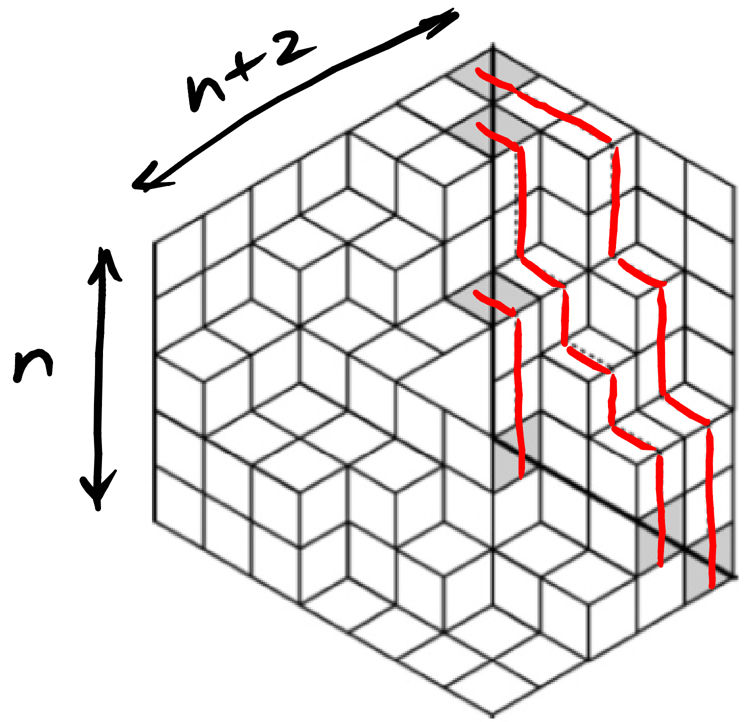




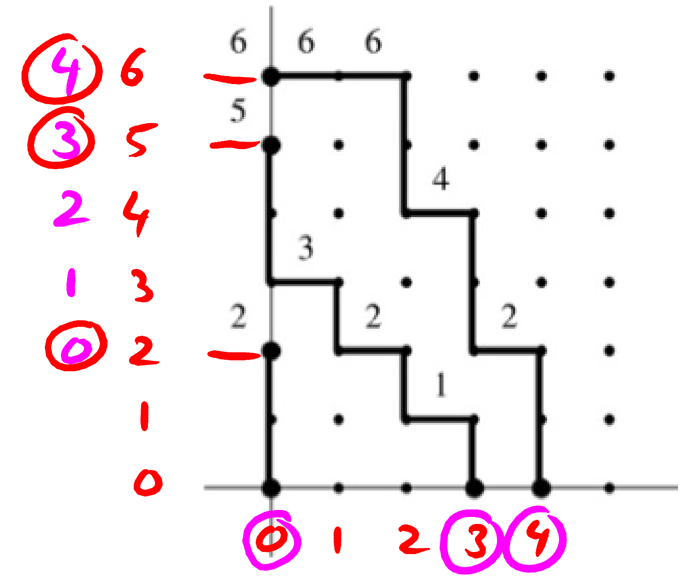
Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole $\leftrightarrow$ Lattice Paths

From DPP to Lattice Paths

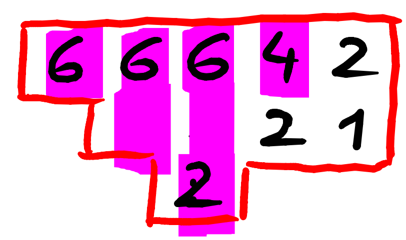
(Krattenthaler '06)

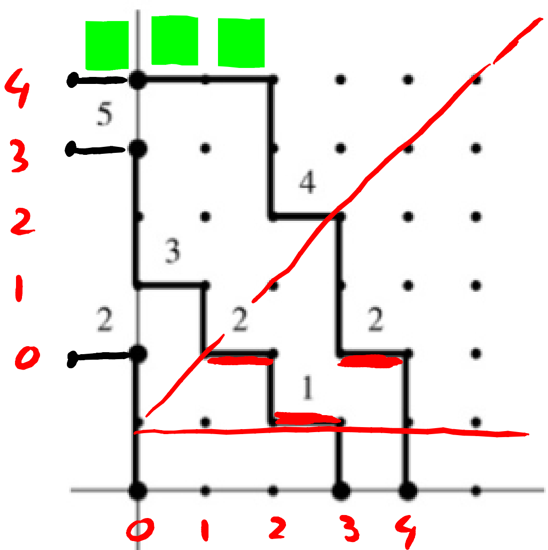


$\longleftrightarrow$

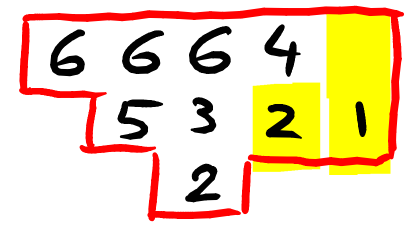


6	6	6	4	2
	5	3	2	1
		2		

Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole $\Leftrightarrow$ Lattice Paths horizontal steps — = non-special parts → 



■ = parts = n

horizontal steps — = special parts → 

} horizontal steps here do not count

Lemma $\det_{n\times n}(I+M) = \sum_{1\le i_1<\dots<i_k\le n} \det\left(M^{i_1\dots i_k}_{i_1\dots i_k}\right)$

M's minor with rows $i_1\dots i_k$, cols $i_1\dots i_k$

Here: M_{ij} = Part. fctn (path $(i,0)\to(0,j)$)

By Gessel–Viennot theorem

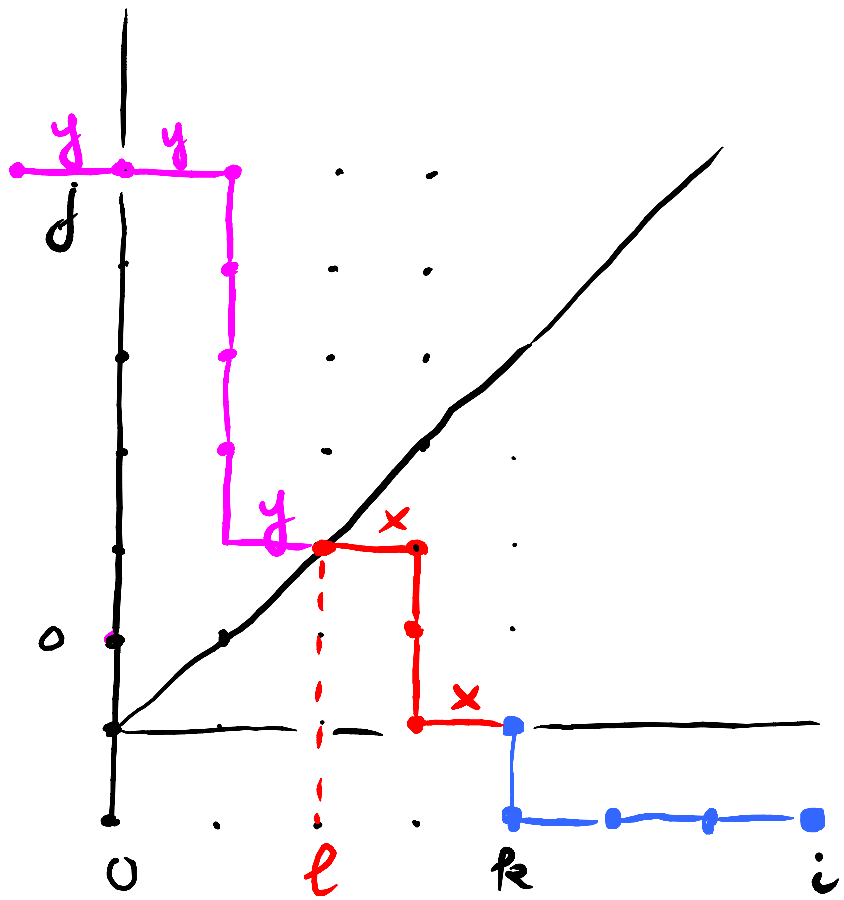
$\det\left(M^{i_1 \dots i_k}_{i_1 \dots i_k}\right)$ = Partition fctn for families of k non-intersecting paths starting at $(i_1,0) \dots (i_k,0)$, ending at $(0,i_1) \dots (0,i_k)$

Partition Function for 1 path :

$$M_{ij} = \sum_{\text{paths } (i0)\to(0j)} x^{\#(-)} \, y^{\#(-)} \quad \left(z^{\#(\overset{n}{-})}\right)$$

horizontal steps : lower wedge (for x), upper wedge (for y)

$z = 1$ for simplicity



$$Z^{(n)}_{DPP}(x, y) = \det(I + M)$$

per special part ↑ (x) ↑ (y) per non-special part

$$M_{i,j} = \sum_{k=0}^{i} \sum_{\ell \geq 0} \binom{k}{\ell} x^{k-\ell} \binom{j+1}{\ell} y^{\ell+1}$$

Generating function:

$$f_{DPP}(z,w) = \sum_{i,j \geq 0} z^i w^j \, (I+M)_{i,j}$$

THM

$$f_{DPP}(z,w) = \frac{1}{1-zw} + \frac{1}{1-z} \frac{yz}{1-xz-w-(y-x)zw}$$

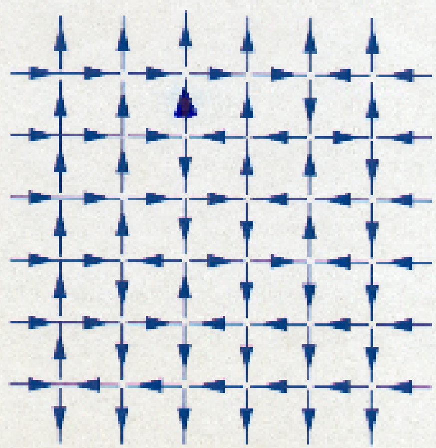
weights: x / special part y / non-special part

ASM

From ASM to 6 Vertex model with Domain Wall Boundary conditions (Kuperberg)

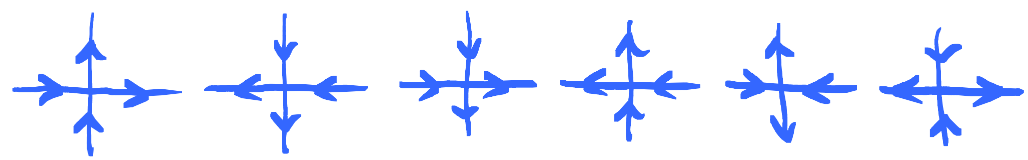
$n \times n$ ASM $\longleftrightarrow$ 6V + DWBC on $n \times n$ grid

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Bijection:

0	0	0	0	1	-1

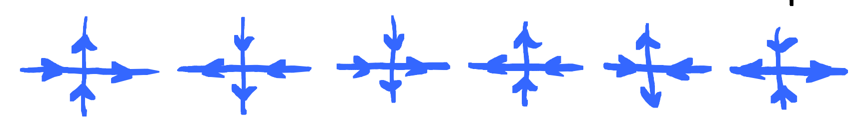


a_1	a_2	b_1	b_2	c_1	c_2

a_1, a_2: $qz - q^{-1}w$; b_1, b_2: $q^{-1}z - qw$; c_1, c_2: $(q^2 - q^{-2})\sqrt{zw}$

(integrable weights)

Refinements :

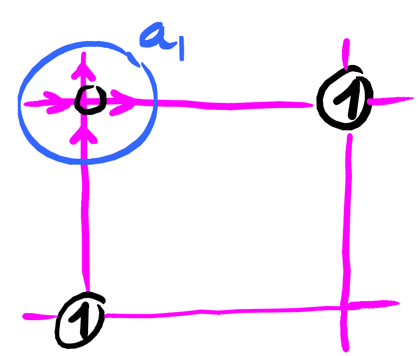


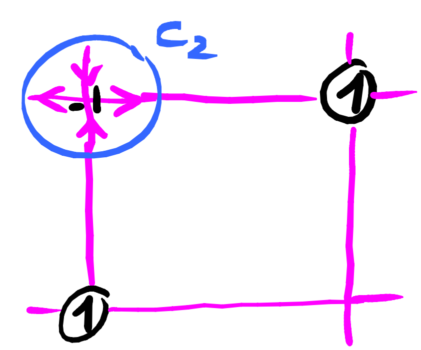
by symmetry :

$$\begin{cases} N_{a_1} = N_{a_2} = \frac{N_a}{2} & N_{b_1} = N_{b_2} = \frac{N_b}{2} \\ N_{c_1} = N_{c_2} + n & N_c = N_{c_1} + N_{c_2} \end{cases}$$

- $\#(-1) = N_{c_2} = \frac{N_c - n}{2}$
- $Inv(A) = N_{a_1} + N_{c_2}$

$\Rightarrow Inv(A) - \#(-1) = N_{a_1} = \frac{N_a}{2}$





Partition function

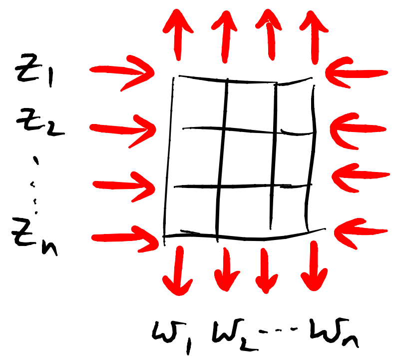
$$Z_{ASM}^{(n)}(x, y, z) = \sum_{\substack{\text{configs of } n\times n \\ 6V\ DWBC}} x^{\frac{N_c - n}{2}} \, y^{\frac{N_a}{2}} \, z^{N_{a'}}, \qquad z = 1$$

usually, one considers

$$Z_{6V}^{(n)}(a,b,c) = \sum_{\substack{\text{configs} \\ 6V\ DWBC}} a^{N_a} b^{N_b} c^{N_c} \times \underbrace{\left(\frac{a'}{a}\frac{b}{b'}\right)^{N_{a'}}}_{=1}$$

$$Z_{6V}^{(n)}(a,b,c) = b^{n^2}\left(\frac{c}{b}\right)^n \sum \left(\frac{c}{b}\right)^{N_c - n} \left(\frac{a}{b}\right)^{N_a}$$

$$\Longleftrightarrow \quad \boxed{x = \left(\frac{c}{b}\right)^2 \qquad y = \left(\frac{a}{b}\right)^2}$$



Partition function of 6V+DWBC

$$Z_n = \frac{\displaystyle\sum_{\substack{\text{configs}\\ \text{on grid}}} \prod_{\substack{\text{vertices}\\ (i\,j)}} \text{weights}(z_i, w_j)}{\displaystyle\hat{\prod_i} c(z_i, w_i)}$$

THM

$$Z_n = \frac{\prod_{i,j} a(z_i,w_j)\, b(z_i,w_j)}{\prod_{i<j} (z_i - z_j)(w_i - w_j)} \det \left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}_{1 \le i,j \le n}$$

(Izergin - Korepin)

recursion relation + symmetries (from commutation of Transfer matrices).

Homogeneous limit: $\begin{cases} z_i \longrightarrow r & \forall i \\ w_j \longrightarrow r^{-1} & \forall j \end{cases}$

$$a(z_i, w_j) \longrightarrow qr - q^{-1}r^{-1} = a(r, r^{-1})$$

$$b(z_i, w_j) \longrightarrow q^{-1}r - qr^{-1} = b(r, r^{-1})$$

$$c(z_i, w_j) \longrightarrow q^2 - q^{-2} = c(r, r^{-1})$$

$$Z_n(q,r) = \frac{(ab)^{n^2}}{c^n} \det_{0 \le i,j \le n-1}\left(\frac{\left(\frac{d}{du}\right)^i}{i!} \frac{\left(\frac{d}{dv}\right)^j}{j!} \left[\frac{c(u,v)}{a(u,v)\,b(u,v)} \right] \Bigg|_{\substack{u=r \\ v=r^{-1}}} \right)$$

↑ by Taylor expansion around the limit

Note: $$\frac{c}{a(u^{-1}v)\,b(u^{-1}v)} = \frac{1}{uv - q^{-2}} - \frac{1}{uv - q^{2}}$$

Taylor-expand:

Define: $$(A_+)_{ij} = \frac{\left(\frac{d}{du}\right)^i}{i!}\frac{\left(\frac{d}{dv}\right)^j}{j!}\frac{1}{uv-q^2}\Bigg|_{\substack{u=r^{-1}\\ v=r^{-1}}} \qquad (A_-)_{ij} = \text{idem}\,(q \to q^{-1})$$

Introduce upper triangular matrix $U(\alpha,\beta)$

$$U(\alpha,\beta)_{i,j} = \begin{cases} \binom{j}{i}\alpha^i\beta^j & \text{if } i \le j \\ 0 & \text{if } i > j \end{cases} \qquad \begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 & \cdots \\ 0 & \alpha\beta & 2\alpha\beta^2 & 3\alpha\beta^3 & \cdots \\ 0 & 0 & \alpha^2\beta^2 & 3\alpha^2\beta^3 & \\ 0 & 0 & 0 & \alpha^3\beta^3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$i,j \in \mathbb{Z}_+ \qquad \alpha,\beta \in \mathbb{C}^*$

THM

$$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1}\bar{U}(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\quad \beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\quad \alpha' = -q^2r^2\beta$ $\quad \beta' = -\frac{1}{\alpha}$

Proof:

1. generating function $(\bar{U}(\alpha\beta)) = \dfrac{1}{1-\beta w(1+\alpha z)}$
2. $U(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$
3. generating function $(A_+) = \dfrac{1}{(r^{-1}+z)(r^{-1}+w) - q^2}$

Holds true for any finite truncation to $i,j \in [0, n-1]$

set $V = U^t(\alpha, \beta)$ $\qquad \tilde{U} = U(\alpha', \beta')$

$\bar{V} = V(q \to q^{-1})$ $\qquad \bar{U} = U(q \to q^{-1})$

then:

$$A_+ = \frac{1}{q^2 - r^{-2}} V^{-1} U$$

$$A_- = \frac{1}{q^{-2} - r^{-2}} \bar{V}^{-1} \bar{U}$$

$$\det(A_- - A_+) = \det(A_-) \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} \tilde{U}^{-1} \bar{V} V^{-1} U \right]$$

$$\propto \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} (\bar{V} V^{-1})(U \bar{U}^{-1}) \right]$$

$U \bar{U}^{-1} = U(-1, 1)$

$\bar{V} V^{-1} = U^t(-y, x)$

Collecting all prefactors, we get:

$$Z^{(n)}_{ASM}(x,y) = \det((1-\nu)I + \nu G)$$

$$\nu = \frac{r^{-2}-q^{-2}}{q^{2}-q^{-2}} \qquad G = U^t(-y,x)\,U(-1,1)$$

$$G_{ij} = \sum_{k\geq 0} \binom{i}{k} y^k \binom{j}{k} x^{i-k}$$

$\Rightarrow$ generating function of $(1-\nu)I + \nu G$ $= f_{ASM}(z,w)$

THM

$$f_{ASM}(z,w) = \frac{1-\nu}{1-zw} + \frac{\nu}{1-zx-w-(y-x)zw}$$

Final identity:

$$Z^{(n)}_{DPP} = \det(I + M) = \det((1-\nu)I + \nu G) = Z^{(n)}_{ASM}$$

Proof:

note that

$$(1-z)(1-(1-\nu)w)\, f_{DPP}(z,w) - \left(1-\frac{z}{1-\nu}\right)(1-w)\, f_{ASM}(z,w)$$

$$= (\underbrace{x\nu(1-\nu) - y(1-\nu) - \nu}_{=0}) \times \text{rational fraction}(z,w)$$

$$\left(x = \left(\frac{q^2-q^{-2}}{q^{-1}r-qr^{-1}}\right)^2,\ y = \left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^2,\ \nu = \left(\frac{r^{-2}-q^{-2}}{q^2-q^{-2}}\right),\ 1-\nu = \left(\frac{q^2-r^{-2}}{q^2-q^{-2}}\right) \right)$$

+ **Remark**: let $A = (A_{ij})_{i,j \geq 0}$ $F = \sum_{i,j \geq 0} A_{ij} z^i w^j$

then $(1 - \lambda z)(1 - \mu w) F(z, w)$ is the generating function for $(I - \lambda S) A (I - \mu S^t)$ where

$S_{i,j} = \delta_{i,j+1}$ "shift" matrix strictly lower triangular $\Rightarrow$ the determinant is unchanged.

Proof completed!

CONCLUSION

MRR PROVED → method of generating fctns for the matrix of which we take the det

→ Bijection ASM – DPP ? (– TSSCPP ?)

→ More refinements ?

TO DO LIST

1. Write the paper
2. Generalizations : DPP with symmetries ⟷ ASM with symmetries

REMARK : integrability of 1+1D Lorentzian gravity v/s integrability of the 6V model

Critical varieties

- $\varphi(g,a) = \dfrac{1-g^2(1-a^2)}{ga}$
- $2\Delta(a,b,c) = \dfrac{a^2+b^2-c^2}{ab} = \dfrac{1+y-x}{\sqrt{y}}$

REMARK : integrability of 1+1D Lorentzian gravity v/s integrability of the 6V model

Critical varieties

- $\varphi(g,a) = \dfrac{1-g^2(1-a^2)}{ga}$
- $2\Delta(a,b,c) = \dfrac{a^2+b^2-c^2}{ab} = \dfrac{1+y-x}{\sqrt{y}}$

same!

$y = g^2a^2 \quad x = g^2$

IT'S A SMALL WORLD!