Bruhat interval polytopes and their friends

Lauren K. Williams, Harvard

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Outline of the talk

• Bruhat interval polytopes and their faces
• Combinatorics of Bruhat interval polytopes
• Some open problems
• Where did Bruhat interval polytopes come from?
• Connection to tropical geometry and (flag) matroids
• Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanuel Tsukerman, and with Jon Boretsky and Chris Eur
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The weak and strong Bruhat orders on $S_n$

Let $S_n$ denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.

Let $\ell(u)$ denote the length of $u$, i.e. the number of inversions of $u$.

The **weak** Bruhat order on $S_n$ is generated by

$$u \leq_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i + 1).$$

The **strong** Bruhat order on $S_n$ is generated by

$$u \leq_t t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

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**Definition (Kodama - W.)**

Let $S_n$ be the symmetric group on $n$ letters. Let $u \leq v$ in the (strong) Bruhat order on $S_n$. The Bruhat interval polytope $Q_{u,v}$ is

$$Q_{u,v} = \text{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}.$$ 

**Rk:** if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $Q_{u,v}$ is the permutohedron.

**Prop. (K.W.):** $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).
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Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.

Open problem

Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)
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**Theorem (Christian Gaetz)**

Let $e$ be the identity permutation. For any $v$, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e,v]$.)
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The dimension of Bruhat interval polytopes

**Def/Lemma:** Let $u \leq v$ in $S_n$, and let $C : u = x(0) \preceq x(1) \ldots \preceq x(\ell) = v$ be any maximal chain in $[u, v]$. Label each edge of $C$ by the transposition $(ab)$ indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $C$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $C$. 

![Diagram of Bruhat interval polytopes with labels and edges indicating the transpositions and equivalence classes.]
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Diagram:
The dimension of Bruhat interval polytopes

**Theorem (Tsukerman-W.)**

The dimension $\dim Q_{u,v}$ of the Bruhat interval polytope $Q_{u,v}$ is

$$\dim Q_{u,v} = n - \#B_{u,v}.$$ 

The equations defining the affine span of $Q_{u,v}$ are

$$\sum_{i \in B_j^+} x_i = \sum_{i \in B_j^+} u_i (= \sum_{i \in B_j^+} v_i), \quad j = 1, 2, \ldots, \#B_{u,v}.$$
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Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \text{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.

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- Arose in our study of full Kostant-Toda lattice (Kodama-W.)

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Choose $u \leq v$ in $W$, where $v$ is a min-length coset rep in $W/W_J$. The (generalized) Bruhat interval polytope for $G/P$ is

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When $G = \text{SL}_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

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- Let $P = P_J$, a parabolic subgroup of $G$. Let $W_J$ be corresponding parabolic subgroup of $W$.
- $\rho_J$ – sum of fund. weights corresp. to $J$, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

**Definition (Tsukerman-W.)**

Choose $u \leq v$ in $W$, where $v$ is a min-length coset rep in $W/W_J$. The (generalized) Bruhat interval polytope for $G/P$ is

$$\tilde{Q}_{u,v}^J := \text{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset t^*_\mathbb{R}.$$ 

When $G = \text{SL}_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}_{w_0v^{-1},w_0u^{-1}}^\emptyset,$$

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Bruhat interval polytopes and their friends
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Connection to tropical geometry and matroids

- **Tropical geometry** is a version of algebraic geometry where we replace polynomials (defined using $\times$, $+$) with tropical polynomials (defined using $+$, $\min$).

- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).

- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow \text{trop Grassmannian}^1$
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Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

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General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real height $h(v)$ to each vertex $v \in \text{Vert}(P)$.
- Consider the polytope $\tilde{P} := \text{Conv}\{(v, h(v)) \mid v \in \text{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the lower faces of $\tilde{P}$ back down to $P$.

Gives polyhedral subdivision of $P$ called coherent (or regular) subdivision.

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Definition/Theorem (Boretsky)

The positive tropical flag variety $\text{Tr Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subset [n]) \in \mathbb{R}^{2^n}$ satisfying positive trop 3-term Plücker relations:

- for $i < j < k < \ell$ and $S$ disjoint from them,
  $$\mu_{Sk} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$

- for $i < j < k$ and $S$ disjoint from them,
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Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.

- Boretsky's result extends to $\text{Tr Fl}_n^{\geq 0}$.
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Polyhedral/ tropical geometry of complete flag positroids

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The positive tropical flag variety $\text{Tr Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subseteq [n]) \in \mathbb{R}^{2^n}$ satisfying positive trop 3-term Plücker relations:

- for $i < j < k < \ell$ and $S$ disjoint from them,
  $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.

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Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.

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Def/Thm (Boretsky): \( \text{Tr Fl}_n^{>0} \) is the set of points \( \mu = (\mu_I \mid I \subset [n]) \in \mathbb{R}^{2^n} \) such that for \( i < j < k < \ell \) and \( S \) disjoint from them,

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Theorem (Joswig-Loho-Luber-Olarte)

Let \( \mu = (\mu_I \mid I \subset [n]) \in \mathbb{R}^{2^n} \). The following are equivalent.

- \( \mu \) lies in the positive tropical complete flag variety \( \text{Tr Fl}_n^{>0} \)
- Every face in the coherent subdivision of \( \text{Perm}_n \) induced by \( \mu \) is a BIP.

Example: \( n = 3 \), \( (\mu_I) \in \mathbb{R}^3 \) such that

\( \mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12} \). Get:

\[
(1, 3, 2) \quad (2, 3, 1) \quad (3, 1, 2) \quad (3, 2, 1) \quad (2, 1, 3)
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Polyhedral/ tropical geometry of complete flag positroids

**Definition/ Theorem (Boretsky):** \( \text{Tr Fl}_n^{>0} \) is the set of points \( \mu = (\mu_I \mid I \subset [n]) \in \mathbb{R}^{2^n} \) such that for \( i < j < k < \ell \) and \( S \) disjoint from them,

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### Subdivisions of Perm\(_4\) into BIPs using Tr Fl\(\mathbf{1}_4^{>0}\)

<table>
<thead>
<tr>
<th>Height function ((\mu_1, \mu_2, \mu_3, \mu_4; \mu_{12}, \mu_{13}, \mu_{14}, \mu_{23}, \mu_{24}, \mu_{34}; \mu_{123}, \mu_{124}, \mu_{134}, \mu_{234}))</th>
<th>Bruhat interval polytopes in subdivision</th>
<th>(f)-vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>((15, -1, -7, -7; -4, -2, -2, -2, -2, -4; 7, -7, -7, -1, 15))</td>
<td>(\mathcal{Q}<em>{3124, 4321}, \mathcal{Q}</em>{3124, 4231}, \mathcal{Q}<em>{2314, 3421}, \mathcal{Q}</em>{2134, 3241}, \mathcal{Q}<em>{1324, 2431}, \mathcal{Q}</em>{1234, 2341})</td>
<td>(24, 46, 29, 6)</td>
</tr>
<tr>
<td>((15, 3, -9, -9; 4, -8, -8, -4, -4, -4, 20, -1, -1, -1, -1, 3))</td>
<td>(\mathcal{Q}<em>{2413, 4321}, \mathcal{Q}</em>{3124, 4231}, \mathcal{Q}<em>{2314, 4231}, \mathcal{Q}</em>{2134, 3241}, \mathcal{Q}<em>{1324, 2431}, \mathcal{Q}</em>{1234, 2341})</td>
<td></td>
</tr>
<tr>
<td>((15, -7, -1, -7; -2, 4, -2, -2, 4, -2, -7, -1, -7, 15))</td>
<td>(\mathcal{Q}<em>{3142, 4321}, \mathcal{Q}</em>{3124, 4312}, \mathcal{Q}<em>{2143, 3421}, \mathcal{Q}</em>{2134, 3412}, \mathcal{Q}<em>{1243, 2431}, \mathcal{Q}</em>{1234, 2413})</td>
<td></td>
</tr>
<tr>
<td>((-1, -1, -1, 3; 4, -8, -4, -8, -4, -8, -4, 20; 15, 3, -9, -9))</td>
<td>(\mathcal{Q}<em>{2413, 4321}, \mathcal{Q}</em>{1423, 4231}, \mathcal{Q}<em>{1324, 4231}, \mathcal{Q}</em>{1324, 4132}, \mathcal{Q}_{1324, 4123})</td>
<td></td>
</tr>
<tr>
<td>((-7, -7, -1, 15; 4, -2, -2, -2, -2, 4; 15, -1, -7, -7))</td>
<td>(\mathcal{Q}<em>{1432, 4321}, \mathcal{Q}</em>{1423, 4312}, \mathcal{Q}<em>{1324, 4231}, \mathcal{Q}</em>{1324, 4132}, \mathcal{Q}_{1234, 4123})</td>
<td></td>
</tr>
<tr>
<td>((-1, -7, -7, 15; -2, -2, -2, 4, -2, -2, 15, -7, -7, -1))</td>
<td>(\mathcal{Q}<em>{3142, 4321}, \mathcal{Q}</em>{2143, 4312}, \mathcal{Q}<em>{2134, 4231}, \mathcal{Q}</em>{1324, 3421}, \mathcal{Q}_{1234, 2413})</td>
<td></td>
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<tr>
<td>((-9, -9, 3, 15; 20, -4, -8, -4, -8, 4, 3, -1, -1, -1))</td>
<td>(\mathcal{Q}<em>{1432, 4321}, \mathcal{Q}</em>{1423, 4312}, \mathcal{Q}<em>{1324, 4231}, \mathcal{Q}</em>{1324, 4132}, \mathcal{Q}_{1324, 4123})</td>
<td></td>
</tr>
<tr>
<td>((11, -7, -7, 3; -6, -6, 4, 4, 2, 2; 11, -7, -7, 3))</td>
<td>(\mathcal{Q}<em>{3142, 4321}, \mathcal{Q}</em>{2143, 4312}, \mathcal{Q}<em>{2134, 4231}, \mathcal{Q}</em>{2134, 4132}, \mathcal{Q}_{1234, 2413})</td>
<td></td>
</tr>
<tr>
<td>((3, 3, -3, -3; 20, -10, -10, -10, -10, 20; -3, -3, 3, 3))</td>
<td>(\mathcal{Q}<em>{2413, 4321}, \mathcal{Q}</em>{3124, 4231}, \mathcal{Q}<em>{3214, 4231}, \mathcal{Q}</em>{1324, 2431}, \mathcal{Q}_{1234, 2431})</td>
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<td>((3, -7, -7, 11; 2, 2, 4, 4, -6, -6, 3, -7, -7, 11))</td>
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<td>((11, -1, -7, -3; -2, -8, -4, -4, 0, 18; 11, -1, -7, -3))</td>
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The flag variety

Let \( R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\} \). The flag variety \( \text{Fl}_{R;n} \) is

\[
\text{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \forall i\}
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Special cases:
- If \( R = [n] \): complete flag variety \( \text{Fl}_n \)
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Can represent an element of \( \text{Fl}_{R;n} \) by an \( r_k \times n \) matrix \( A \) such that the span of the top \( r_i \) rows is \( V_i \).

Example: \( A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \) represents an element of \( \text{Fl}_3 \).

For \( I \subset [n] \), the Plücker coordinate (or flag minor) \( p_I(A) \) is the determinant of the submatrix in columns \( I \) and rows 1, 2, \ldots, \( |I| \).
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Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The flag variety $\text{Fl}_{R; n}$ is

$$\text{Fl}_{R; n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety $\text{Fl}_n$
- If $R = \{r\}$: Grassmannian $\text{Gr}_{r,n}$.

Can represent an element of $\text{Fl}_{R; n}$ by an $r_k \times n$ matrix $A$ such that the span of the top $r_i$ rows is $V_i$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of $\text{Fl}_3$.

For $I \subset [n]$, the Plücker coordinate (or flag minor) $p_I(A)$ is the determinant of the submatrix in columns $I$ and rows $1, 2, \ldots, |I|$.
Let $R = (r_1 < \cdots < r_k) \subset [n]$. A flag matroid of ranks $R$ is a sequence $M = (M_1, \ldots, M_k)$ of matroids of ranks $r_i$ on $[n]$ such that all vertices of

$$P(M) = P(M_1) + \cdots + P(M_k)$$

(Minkowski sum) are equidistant from the origin. $P(M)$ then called a flag matroid polytope.

If $\exists r_k \times n$ matrix $A$ whose top $r_i$ rows realize $M_i \ \forall i$, say $A$ realizes $M$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let $A$ be a matrix giving a realization of flag matroid $M = (M_1, \ldots, M_k)$. If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $M$ is a flag positroid and $P(M)$ a flag positroid polytope.

Note: by definition, a flag positroid is realizable.
Let $R = (r_1 < \cdots < r_k) \subset [n]$. A **flag matroid** of ranks $R$ is a sequence $\mathcal{M} = (M_1, \ldots, M_k)$ of matroids of ranks $r_i$ on $[n]$ such that all vertices of $P(\mathcal{M}) = P(M_1) + \cdots + P(M_k)$ (Minkowski sum) are equidistant from the origin. $P(\mathcal{M})$ then called a **flag matroid polytope**.

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Flag matroids, flag positroids, and their polytopes

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Theorem (Boretsky–Eur–W)

Let $R = (a, a + 1, \ldots, b) \subset [n]$. Let
\[ \mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{[n]}. \]

TFAE:

- $\mu$ lies in the nonnegative tropical flag variety $\text{Tr Fl}_{R;n}^{\geq 0}$, that is, $\mu$ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\mu^a) + \cdots + P(\mu^b)$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.\(^2\)

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\(^2\)One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff $R$ a consecutive subset (Bloch–Karp).
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A complete flag matroid polytope is a Bruhat interval polytope iff all its 1 and 2-dimensional faces are.

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Recall: an oriented matroid $\chi_M$ of rank $d$ on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is “consistent w/ 3-term Plücker relations.” Related notions of oriented flag matroid, positively oriented (flag) matroid. Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

**Corollary (Boretsky-Eur-W)**

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it’s an oriented flag matroid.

**Corollary (Boretsky-Eur-W)**

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Recall: an oriented matroid $\chi_M$ of rank $d$ on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is “consistent w/ 3-term Plücker relations.” Related notions of oriented flag matroid, positively oriented (flag) matroid. Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it’s an oriented flag matroid.

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“The full Kostant-Toda hierarchy on the positive flag variety” with Yuji Kodama, Comm Math Phys 2015.

Figure from Nadeau-Tewari 2208.04128 “Remixed Eulerian numbers”
Quick intro to tropical geometry

- Tropical varieties are “piecewise-linear” versions of ordinary varieties.
- Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ be the field of Puiseux series. Each element has the form $ct^a + \ldots$ where $c \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{Q}$, and the other terms have larger exponents.
- Define valuation $\text{val}(ct^a + \ldots) = a$.
- For $I \subset \mathcal{C}[x_1, \ldots, x_n]$ an ideal, the corresponding tropical variety is $\text{Trop} V(I) := \text{val}(V(I) \cap (\mathcal{C} \setminus \{0\})^n)$. That is, we compute the “ordinary” variety over the field of Puiseux series, but then apply the valuation to each coordinate.
- The positive Puiseux series $\mathcal{C}^+$ consists of $ct^a + \ldots$ where $c \in \mathbb{R}_{>0}$.
- The positive tropical variety is $\text{Trop}^+ V(I) := \text{val}(V(I) \cap (\mathcal{C}^+ \setminus \{0\})^n)$.
- Note: If $f(t), g(t) \in \mathcal{C}^+$, then $\text{val}(f(t)g(t)) = \text{val}(f(t) + g(t))$ and $\text{val}(f(t) + g(t)) = \min(\text{val}(f(t)), \text{val}(g(t)))$. 
Thm (Lukowski-Parisi-W, Speyer-W, Arkani-Hamed-Lam-Spradlin)

Let $\mu = (\mu_I \mid I \in \binom{[n]}{r}) \in \mathbb{R}^{\binom{[n]}{r}}$. The following are equivalent.

- $\mu$ lies in the positive tropical Grassmannian $\text{Tr} \text{Gr}_{r,n}^>^0$ (the closure of the coordinate-wise valuation of $\text{Gr}_{r,n}$ over positive Puiseux series)
- $\mu$ obeys the positive tropical 3-term Plücker relations: for $i < j < k < \ell$ and $S$ disjoint from them, $|S| = r - 2$,
  $$\mu_{Si} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Sil} + \mu_{Sjk}).$$
- Every face in the regular (coherent) subdivision of the hypersimplex $\Delta_{r,n} = \text{Conv}(e_I \mid I \in \binom{[n]}{r})$ induced by $\mu$ is a positroid polytope.

Example: $n = 4, r = 2, (\mu_I) \in \mathbb{R}^{\binom{[4]}{2}}$ such that

$$\mu_{13} + \mu_{24} = \mu_{23} + \mu_{14} < \mu_{12} + \mu_{34}.$$ Get:
The lifting property and a generalization

**Lifting property:** Suppose $u < v$ in Bruhat order and $s$ is a simple reflection such that $vs < v$ and $us > u$. Then $u \leq vs < v$ and $u < us \leq v$.

**Caution:** such an $s$ may not exist.

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**Def:** Say a transposition $(ij)$ is *inversion-minimal* on $(u, v)$ if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

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**Theorem (T.W.) - Generalized lifting property**

Suppose $u < v$ in Bruhat order on $S_n$. Choose a transposition $(ij)$ which is inversion-minimal on $(u, v)$; one always exists. Then $u \leq v(ij) < v$ and $u < u(ij) \leq v$. 
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\[ \begin{array}{c}
\text{v(ij)} \\
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**Theorem (T.W.) - Generalized lifting property**

Suppose $u < v$ in the Bruhat order on $S_n$. Choose a transposition $t = (ij)$ which is inversion-minimal on $(u, v)$; one always exists. Then $u \leq v(ij) \preccurlyeq v$ and $u \preccurlyeq u(ij) \leq v$.

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Let $u = 2143$, $v = 3241$, and $t = (24)(12)$.

```
  v = 3241
     /   \
   /     \
 t = (24)   (12)
     \
 3142   2341
     /   \
   /     \
 (14)   t = (24)
     \
 u = 2143
```
Generalization of the recurrence for $R$-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced $R$-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \# \mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that $R$-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0$, if $u \not\leq v$.
2. $R_{u,v}(q) = 1$, if $u = v$.
3. If $vs \leq v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us, vs}(q) & \text{if } us \prec u, \\ qR_{us, vs}(q) + (q - 1)R_{u, vs}(q) & \text{if } us \succ u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on $(u, v)$. Then $R_{u,v}(q) = qR_{ut, vt}(q) + (q - 1)R_{u, vt}(q)$. 

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