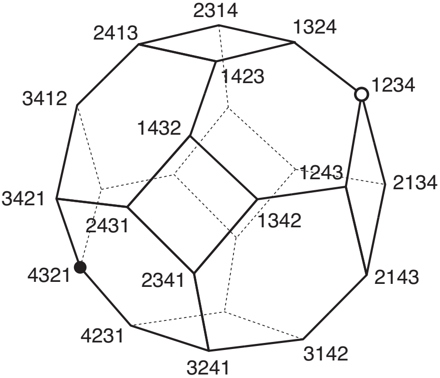
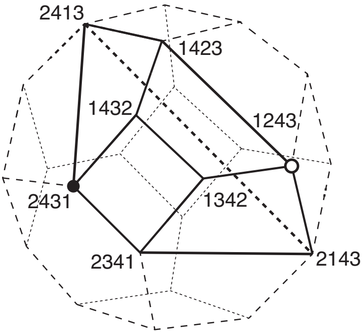
Bruhat interval polytopes and their friends

Lauren K. Williams, Harvard





Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,

and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,
and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,
and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,

and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,
and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,

and with Jon Boretsky and Chris Eur

Outline of the talk

- Bruhat interval polytopes and their faces
- Combinatorics of Bruhat interval polytopes
- Some open problems
- Where did Bruhat interval polytopes come from?
- Connection to tropical geometry and (flag) matroids
- Subdivisions of the permutohedron

Based on: joint works with Yuji Kodama, Emmanual Tsukerman,

and with Jon Boretsky and Chris Eur

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.
Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.

Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.

Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.
Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.
Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, i.e. the set of all permutations $u = (u_1, \ldots, u_n)$ of $[n]$.
Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

The weak and strong Bruhat orders on S_n

Let S_n denote the symmetric group on $[n] = \{1, 2, \dots, n\}$, i.e. the set of all permutations $u = (u_1, \dots, u_n)$ of $[n]$.
Let $\ell(u)$ denote the length of u, i.e. the number of inversions of u.

The weak Bruhat order on S_n is generated by

$$u \lessdot_w s_i u \text{ if } \ell(s_i u) = \ell(u) + 1, \text{ where } s_i = (i, i+1).$$

The strong Bruhat order on S_n is generated by

$$u \lessdot t_{ij} u \text{ if } \ell(t_{ij} u) = \ell(u) + 1, \text{ where } t_{ij} = (i, j).$$

picture!

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $Q_{u,v}$ is $Q_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.

Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $Q_{u,v}$ is the permutohedron.

Prop. (K.W.): $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $Q_{u,v}$ is $Q_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.

Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $Q_{u,v}$ is the permutohedron.

Prop. (K.W.): $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $Q_{u,v}$ is
$Q_{u,v} = \text{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.

Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $Q_{u,v}$ is the permutohedron.

Prop. (K.W.): $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $Q_{u,v}$ is

$Q_{u,v} = \text{Conv}\{(z(1), \dots, z(n)) \mid u \leq z \leq v\}$.

Rk: if $u = e$ and $v = w_0 = (n, n-1, \dots, 1)$, $Q_{u,v}$ is the permutohedron.

Prop. (K.W.): $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

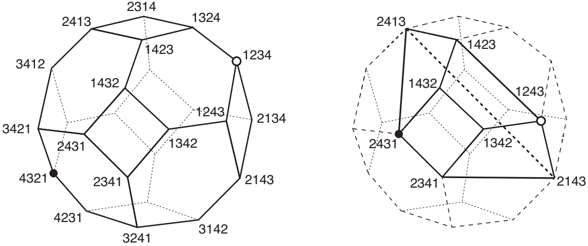
Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is $\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.

Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $\mathrm{Q}_{u,v}$ is the permutohedron.

Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $Q_{u,v}$ is
$Q_{u,v} = \text{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.

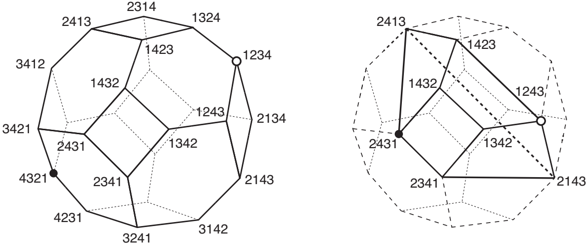


Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $Q_{u,v}$ is the permutohedron.

Prop. (K.W.): $Q_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is $\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.



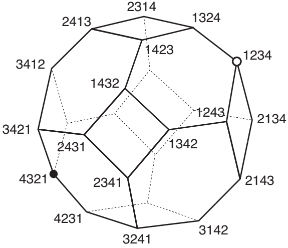
Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $\mathrm{Q}_{u,v}$ is the permutohedron.

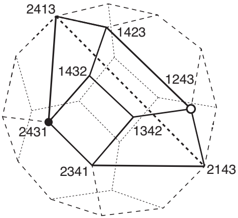
Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is
$\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.





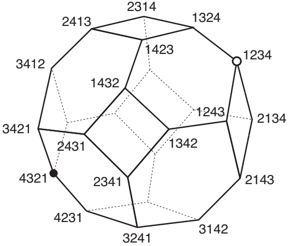
Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $\mathrm{Q}_{u,v}$ is the permutohedron.

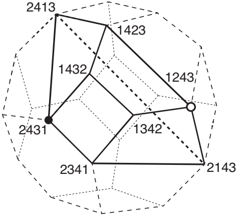
Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Bruhat interval polytopes (BIPs)

Definition (Kodama -W.)

Let S_n be the symmetric group on n letters. Let $u \leq v$ in the (strong) Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is
$\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}$.





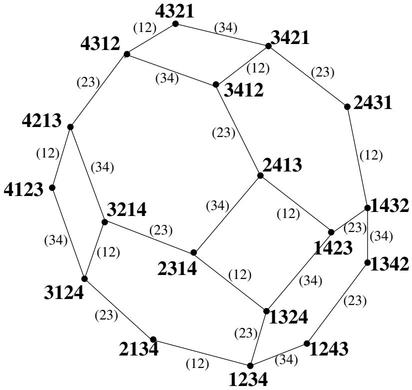
Rk: if $u = e$ and $v = w_0 = (n, n-1, \ldots, 1)$, $\mathrm{Q}_{u,v}$ is the permutohedron.

Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Combinatorics of Bruhat interval polytopes

Facts

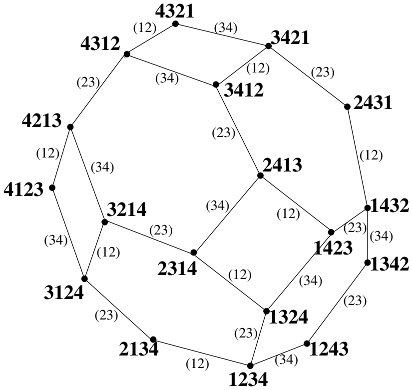
- The 1-skeleton of the permutohedron is the Hasse diagram of the weak Bruhat order.
- Faces of permutohedra $\cong$ products of smaller permutohedra.



Combinatorics of Bruhat interval polytopes

Facts

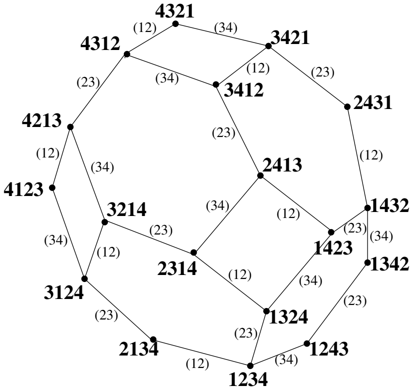
- The 1-skeleton of the permutohedron is the Hasse diagram of the weak Bruhat order.
- Faces of permutohedra ≅ products of smaller permutohedra.



Combinatorics of Bruhat interval polytopes

Facts

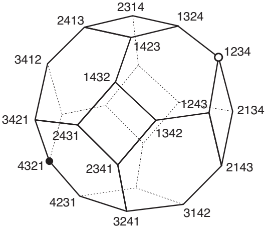
- The 1-skeleton of the permutohedron is the Hasse diagram of the weak Bruhat order.
- Faces of permutohedra $\cong$ products of smaller permutohedra.

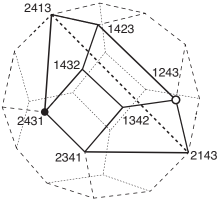


Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.



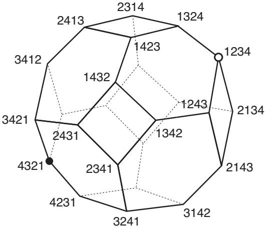


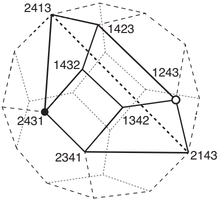
Open problem

Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





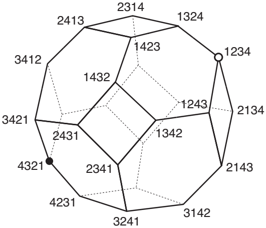
Open problem

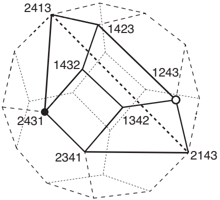
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





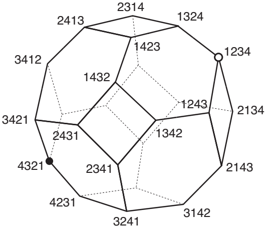
Open problem

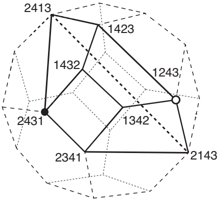
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





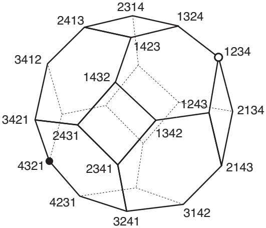
Open problem

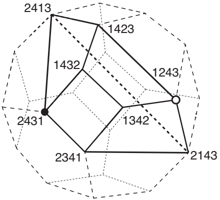
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





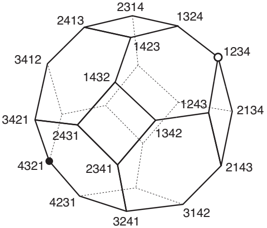
Open problem

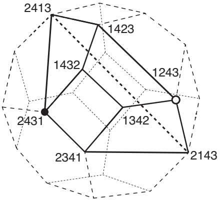
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





Open problem

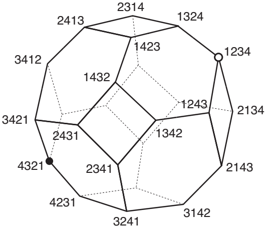
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces?

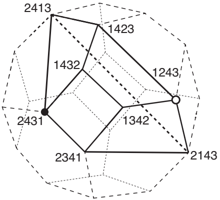
(With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ is a Bruhat interval polytope $Q_{x,y}$ where $u \leq x \leq y \leq v$.





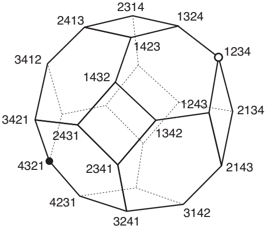
Open problem

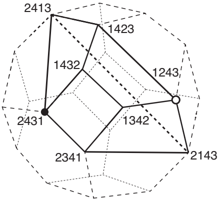
Can one characterize all edges of $Q_{u,v}$? Can one characterize all faces? (With Tsukerman we have a complicated criterion for determining when $Q_{x,y}$ is a face of $Q_{u,v}$; not very nice.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$.





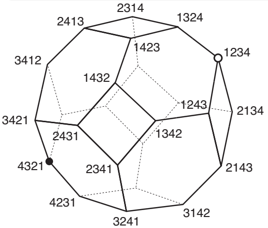
Theorem (Christian Gaetz)

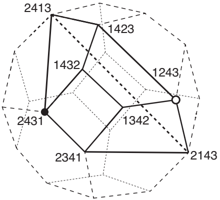
Let e be the identity permutation. For any v, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e, v]$.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$.





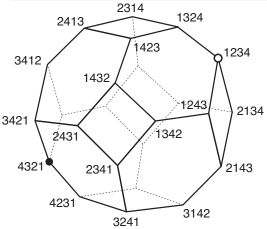
Theorem (Christian Gaetz)

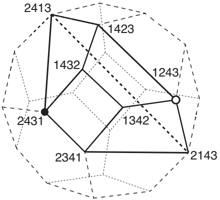
Let e be the identity permutation. For any v, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e, v]$.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$.





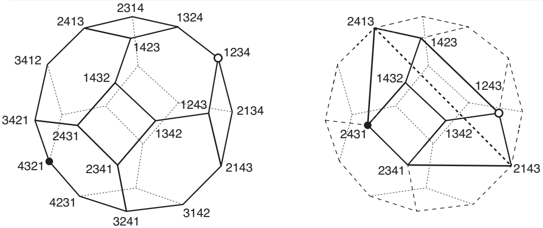
Theorem (Christian Gaetz)

Let e be the identity permutation. For any v, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e, v]$.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$.



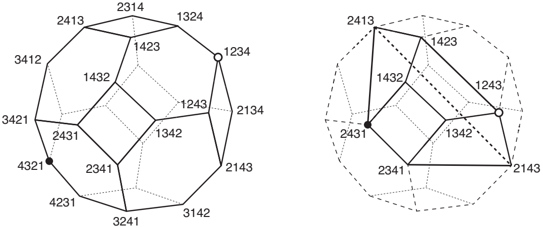
Theorem (Christian Gaetz)

Let e be the identity permutation. For any v, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e, v]$.)

Combinatorics of Bruhat interval polytopes

Theorem (Kodama-W., Tsukerman-W.)

Every edge of a Bruhat interval polytope $Q_{u,v}$ comes from a cover relation in the strong Bruhat order. Moreover, each face of a Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$.

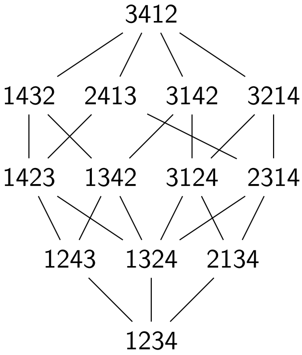


Theorem (Christian Gaetz)

Let e be the identity permutation. For any v, the 1-skeleton of $Q_{e,v}$ is the Hasse diagram of a lattice. (Poset is intermediate in strength between the weak and strong Bruhat order on $[e, v]$.)

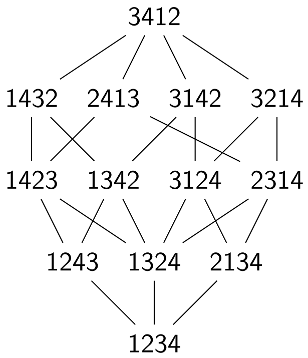
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $C : u = x_{(0)} < x_{(1)} \ldots < x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of C by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on C. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of C.



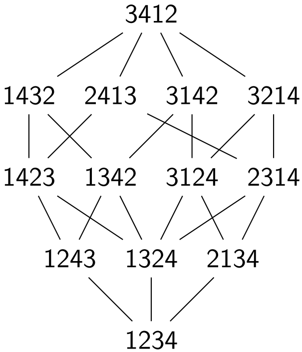
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \dots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \dots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \dots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



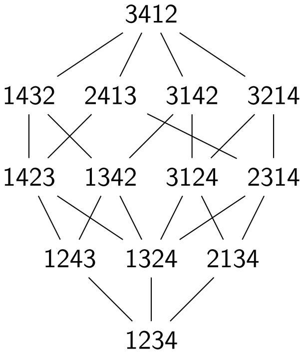
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



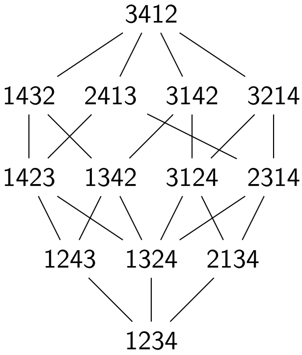
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



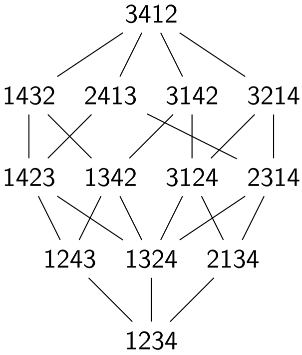
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



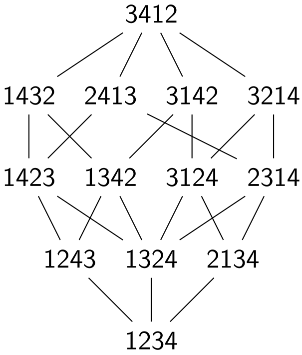
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



The dimension of Bruhat interval polytopes

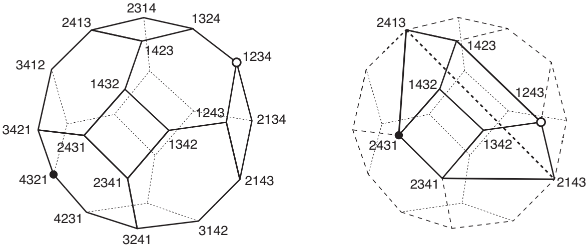
Theorem (Tsukerman-W.)

The dimension $\dim Q_{u,v}$ of the Bruhat interval polytope $Q_{u,v}$ is

$$\dim Q_{u,v} = n - \#B_{u,v}.$$

The equations defining the affine span of $Q_{u,v}$ are

$$\sum_{i \in B^j} x_i = \sum_{i \in B^j} u_i \left(= \sum_{i \in B^j} v_i\right), \quad j = 1, 2, \ldots, \#B_{u,v}.$$



The dimension of Bruhat interval polytopes

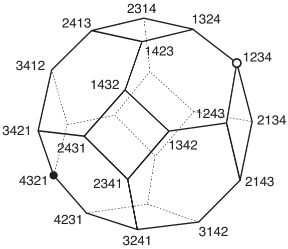
Theorem (Tsukerman-W.)

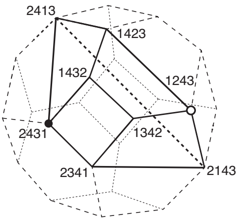
The dimension $\dim \mathrm{Q}_{u,v}$ of the Bruhat interval polytope $\mathrm{Q}_{u,v}$ is

$$\dim \mathrm{Q}_{u,v} = n - \#B_{u,v}.$$

The equations defining the affine span of $\mathrm{Q}_{u,v}$ are

$$\sum_{i \in B^j} x_i = \sum_{i \in B^j} u_i \left(= \sum_{i \in B^j} v_i\right), \quad j = 1, 2, \ldots, \#B_{u,v}.$$



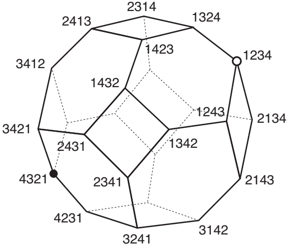


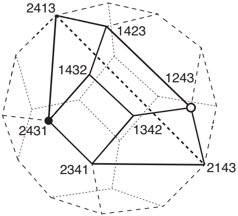
The dimension of Bruhat interval polytopes

Theorem (Tsukerman-W.)

The dimension $\dim Q_{u,v}$ of the Bruhat interval polytope $Q_{u,v}$ is
$$\dim Q_{u,v} = n - \#B_{u,v}.$$

The equations defining the affine span of $Q_{u,v}$ are
$$\sum_{i \in B^j} x_i = \sum_{i \in B^j} u_i (= \sum_{i \in B^j} v_i), \quad j = 1, 2, \ldots, \#B_{u,v}.$$





Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.

(Gelfand-Goresky-MacPherson-Serganova)

Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \text{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.

(Gelfand-Goresky-MacPherson-Serganova)

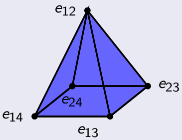
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

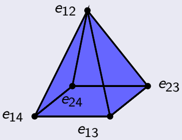
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \text{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

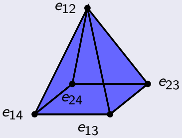
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \ldots, b_r\} \in \mathcal{B}$ iff columns $b_1, \ldots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

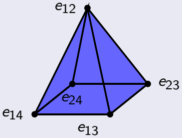
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

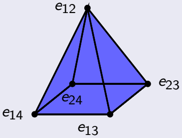
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \text{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

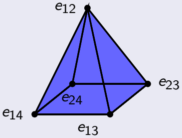
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

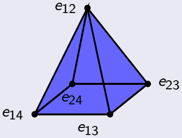
Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \dots, b_r\} \in \mathcal{B}$ iff columns $b_1, \dots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Given a subset $S \subseteq [n]$, let $e_S := \sum_{i \in S} e_i \in \mathbb{R}^n$.

Given a collection $\mathcal{B} \subset \binom{[n]}{d}$, let $P(\mathcal{B}) = \mathrm{Conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.
If every edge of $P(\mathcal{B})$ is parallel to $e_i - e_j$ for some $i \neq j$,
we say $\mathcal{B}$ is the set of *bases* of the *matroid* $M = ([n], \mathcal{B})$.
Let $P(M) := P(\mathcal{B})$. Called *matroid polytope*.



(Gelfand-Goresky-MacPherson-Serganova)

Each full rank $r \times n$ matrix A gives rise to a matroid $M_A = ([n], \mathcal{B})$, where $\{b_1, \ldots, b_r\} \in \mathcal{B}$ iff columns $b_1, \ldots, b_r$ of A are linearly independent.
Such a matroid is called *realizable*. example!

If $M = ([n], \mathcal{B})$ is a matroid and $T \subseteq [n]$, let $r_M(T) = \max_{B \in \mathcal{B}} |\{T \cap B\}|$.
(Called *rank* of T.)

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \dots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$Q_{u,v} = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \ldots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and $Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$\mathrm{Q}_{u,v} = \left\{ \mathrm{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall \, A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \dots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$Q_{u,v} = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u, v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \ldots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$Q_{u,v} = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \forall A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \ldots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$Q_{u,v} = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \,\forall A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \dots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$\mathrm{Q}_{u,v} = \left\{ \mathrm{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \dots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$\mathrm{Q}_{u,v} = \left\{ \mathrm{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall \, A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Connection with matroids and their polytopes

Proposition (Tsukerman-W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n$, define $\mathcal{B}^k$ to be

$$\{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ s.t. } I = \{z^{-1}(n), z^{-1}(n-1), \dots, z^{-1}(n-k+1).\}\}.$$

Then $M^k := ([n], \mathcal{B}^k)$ is a matroid and
$Q_{u,v}$ is the Minkowski sum of matroid polytopes $P(M^1) + \cdots + P(M^n)$.

Corollary: Inequality description of $Q_{u,v}$

$$\mathrm{Q}_{u,v} = \left\{ \mathrm{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{M^j}(A) \, \forall \, A \subset [n] \right\}$$

Question: Is there a shorter description? Characterization of facets?

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the $\#$ of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

Where did Bruhat interval polytopes come from?

- Arose in our study of full Kostant-Toda lattice (Kodama-W.)
- Each Bruhat interval polytope is the moment map image of a Richardson variety $\mathcal{R}_{u,v}$ in the complete flag variety Fl_n (K-W.).
- Kazhdan-Lusztig's R-polynomials $R_{u,v}(q)$ have geom. interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the # of $\mathbb{F}_q$-points in the Richardson variety. Studying BIPs $\rightsquigarrow$ new recurrence for R-polynomials. (T-W.)

The interpretation of BIPs as moment map images of Richardson varieties (subvarieties of the complete flag variety Fl_n) leads to a natural generalization of BIPs ...

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = \mathrm{SL}_n(\mathbb{R})$ w/ fund. weights e_1, $e_1 + e_2$, ..., $e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = SL_n(\mathbb{R})$ w/ fund. weights e_1, $e_1 + e_2$, $\ldots$, $e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. *The (generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \text{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = SL_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \text{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = SL_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = SL_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{Q}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = \mathrm{SL}_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$Q_{u,v} = \tilde{Q}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

- Let G be a semisimple simply connected algebraic group with torus T and Weyl group W.
- Let $P = P_J$, a parabolic subgroup of G. Let W_J be corresponding parabolic subgroup of W.
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.

Definition (Tsukerman-W.)

Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J. The *(generalized) Bruhat interval polytope for G/P* is

$$\tilde{\mathrm{Q}}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

When $G = \mathrm{SL}_n(\mathbb{R})$ w/ fund. weights $e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_{n-1}$, and $J = \emptyset$, we have:

$$\mathrm{Q}_{u,v} = \tilde{\mathrm{Q}}^{\emptyset}_{w_0 v^{-1}, w_0 u^{-1}},$$

where w_0 is the longest permutation.

(Generalized) Bruhat interval polytopes for G/P

(Generalized) Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- *positroid* polytopes

Theorem (T.W.)

The face of a (generalized) Bruhat interval polytopes for G/P is again a (generalized) Bruhat interval polytope for G/P.

(Generalized) Bruhat interval polytopes for G/P

(Generalized) Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- *positroid* polytopes

Theorem (T.W.)

The face of a (generalized) Bruhat interval polytopes for G/P is again a (generalized) Bruhat interval polytope for G/P.

(Generalized) Bruhat interval polytopes for G/P

(Generalized) Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- *positroid* polytopes

Theorem (T.W.)

The face of a (generalized) Bruhat interval polytopes for G/P is again a (generalized) Bruhat interval polytope for G/P.

(Generalized) Bruhat interval polytopes for G/P

(Generalized) Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- *positroid* polytopes

Theorem (T.W.)

The face of a (generalized) Bruhat interval polytopes for G/P is again a (generalized) Bruhat interval polytope for G/P.

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+$, min).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields.
  In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields.
  In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Connection to tropical geometry and matroids

- *Tropical geometry* is a version of algebraic geometry where we replace polynomials (defined using $\times, +$) with tropical polynomials (defined using $+, \min$).
- Various connections between tropical geometry and matroids have been discovered starting around 2008 (Speyer, Herrmann, Jensen, Joswig, Sturmfels ...).
- Baker-Bowler have a framework of matroids over hyperfields. In this framework,
  - Grassmannian over the *tropical hyperfield* $\rightsquigarrow$ *trop Grassmannian*[1]
  - Grassmannian over the *Krasner hyperfield* $\rightsquigarrow$ the set of matroids.
- Jarra-Lorscheid generalized framework to flag variety/ flag matroids.

Next: link our story more closely to tropical geometry and flag variety.
But first: an aside on decompositions of the permutohedron ...

[1] actually the Dressian, which contains the trop Grassmannian

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019:
  "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

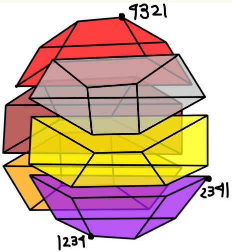


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm$_n$ into BIPs?

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019:
  "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

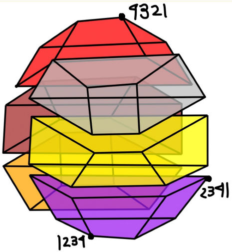


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm_n into BIPs?

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019:
  "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

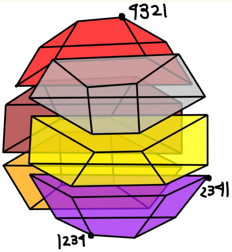


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm_n into BIPs?

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019:
  "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

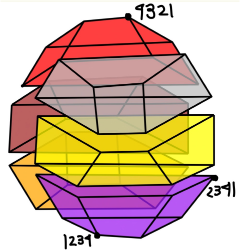


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm_n into BIPs?

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019: "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

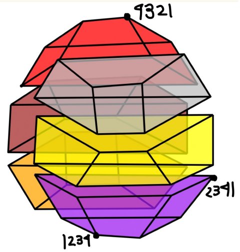


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm_n into BIPs?

Decompositions of the permutohedron

Subdivisions of the permutohedron into special Bruhat interval polytopes (isomorphic to cubes) appeared in recent work of:

- Harada-Horiguchi-Masuda-Park 2019: "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Tsetlin polytope"
- Nadeau-Tewari 2022: "Remixed Eulerian numbers"

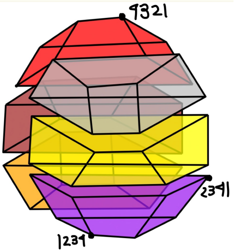


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Can we systematically understand how to cut up Perm_n into BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \text{Vert}(P)$.
- Consider the polytope $\tilde{P} := \text{Conv}\{(v, h(v)) \mid v \in \text{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \text{Vert}(P)$.
- Consider the polytope $\tilde{P} := \text{Conv}\{(v, h(v)) \mid v \in \text{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Coherent subdivisions of polytopes

General method to produce a polyhedral subdivision of a polytope $P \subset \mathbb{R}^d$:

- Assign a real *height* $h(v)$ to each vertex $v \in \mathrm{Vert}(P)$.
- Consider the polytope $\tilde{P} := \mathrm{Conv}\{(v, h(v)) \mid v \in \mathrm{Vert}(P)\} \subset \mathbb{R}^{d+1}$.
- Project the *lower faces* of $\tilde{P}$ back down to P.

Gives polyhedral subdivision of P called *coherent* (or *regular*) subdivision.

example?

Questions:

- Can we produce subdivisons of Perm_n into BIPs using this method?
- When does a coherent subdivision of Perm_n consist of BIPs?

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.
- for $i < j < k$ and S disjoint from them,
  $\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij})$.

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
  $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
  $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
  $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.
- for $i < j < k$ and S disjoint from them,
  $\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij})$.

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
  $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
$$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
$$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\mathrm{Tr}\,\mathrm{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Definition/Theorem (Boretsky)

The *positive tropical flag variety* $\operatorname{Tr}\operatorname{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ satisfying *positive trop 3-term Plücker relations*:

- for $i < j < k < \ell$ and S disjoint from them,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- for $i < j < k$ and S disjoint from them,
  $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

Remarks:

- Is a theorem because original defn of pos trop flag variety is as the closure of the coordinate-wise valuation of the flag variety over positive Puiseux series.
- Boretsky's result extends to $\operatorname{Tr}\operatorname{Fl}_n^{\geq 0}$.

Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\operatorname{Tr}\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.
- $\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij})$.

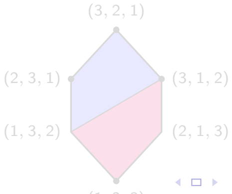
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\operatorname{Tr}\mathrm{Fl}_n^{\geq 0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

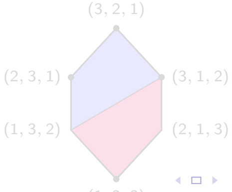
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.
- $\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij})$.

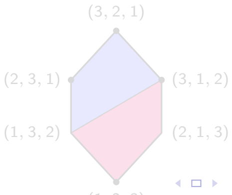
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

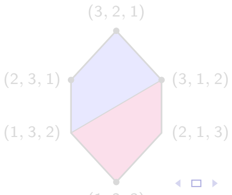
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- $$\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij}).$$

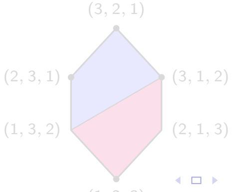
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Polyhedral/ tropical geometry of complete flag positroids

Def/Thm (Boretsky): $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ is the set of points $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$ such that for $i < j < k < \ell$ and S disjoint from them,

- $\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk})$.
- $\mu_{Sj} + \mu_{Sik} = \min(\mu_{Si} + \mu_{Sjk}, \mu_{Sk} + \mu_{Sij})$.

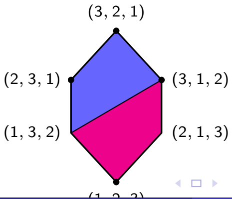
Theorem (Joswig-Loho-Luber-Olarte)

Let $\mu = (\mu_I \mid I \subsetneq [n]) \in \mathbb{R}^{2^n}$. The following are equivalent.

- μ lies in the positive tropical complete flag variety $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$
- Every face in the coherent subdivision of Perm_n induced by μ is a BIP.

Example: $n = 3, (\mu_I) \in \mathbb{R}^{2^3}$ such that

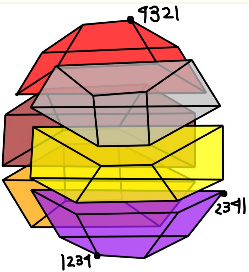
$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$. Get:



Subdivisions of Perm_4 into BIPs using $\mathrm{Tr}\,\mathrm{Fl}_4^{>0}$

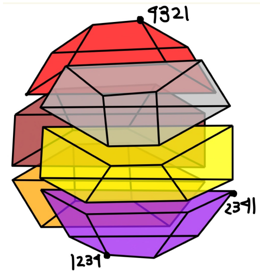
Height function $(\mu_1, \mu_2, \mu_3, \mu_4; \mu_{12}, \mu_{13}, \mu_{14}, \mu_{23}, \mu_{24}, \mu_{34}; \mu_{123}, \mu_{124}, \mu_{134}, \mu_{234})$	Bruhat interval polytopes in subdivision	f-vector
$(15, -1, -7, -7; 4, -2, -2, -2, -2, 4; -7, -7, -1, 15)$	$Q_{3214,4321}, Q_{3124,4231}, Q_{2314,3421}, Q_{2134,3241}, Q_{1324,2431}, Q_{1234,2341}$	(24, 46, 29, 6)
$(15, 3, -9, -9; 4, -8, -8, -4, -4, 20; -1, -1, -1, 3)$	$Q_{2413,4321}, Q_{3124,4231}, Q_{2314,4231}, Q_{2134,3241}, Q_{1324,2431}, Q_{1234,2341}$	
$(15, -7, -1, -7; -2, 4, -2, -2, 4, -2; -7, -1, -7, 15)$	$Q_{3142,4321}, Q_{3124,4312}, Q_{2143,3421}, Q_{2134,3412}, Q_{1243,2431}, Q_{1234,2413}$	
$(-1, -1, -1, 3; 4, -8, -4, -8, -4, 20; 15, 3, -9, -9)$	$Q_{2413,4321}, Q_{1423,4231}, Q_{1342,4231}, Q_{1324,4213}, Q_{1243,4132}, Q_{1234,4123}$	
$(-7, -7, -1, 15; 4, -2, -2, -2, -2, 4; 15, -1, , -7, -7)$	$Q_{1432,4321}, Q_{1423,4312}, Q_{1342,4231}, Q_{1324,4213}, Q_{1243,4132}, Q_{1234,4123}$	
$(-1, -7, -7, 15; -2, -2, 4, 4, -2, -2; 15, -7, -7, -1)$	$Q_{3142,4321}, Q_{2143,4312}, Q_{2134,4213}, Q_{1342,3421}, Q_{1243,3412}, Q_{1234,2413}$	
$(-9, -9, 3, 15; 20, -4, -8, -4, -8, 4; 3, -1, -1, -1)$	$Q_{1432,4321}, Q_{1423,4312}, Q_{1342,4231}, Q_{1324,4213}, Q_{1324,4132}, Q_{1234,3142}$	
$(11, -7, -7, 3; -6, -6, 4, 4, 2, 2; 11, -7, -7, 3)$	$Q_{3142,4321}, Q_{2143,4312}, Q_{2134,4213}, Q_{2143,3421}, Q_{1243,2431}, Q_{1234,2413}$	
$(3, 3, -3, -3; 20, -10, -10, -10, -10, 20; -3, -3, 3, 3)$	$Q_{2413,4321}, Q_{3124,4231}, Q_{2314,4231}, Q_{1324,2431}, Q_{1324,3241}, Q_{1234,3142}$	
$(3, -1, -1, -1; 20, -4, -4, -8, -8, 4; -9, -9, 3, 15)$	$Q_{3214,4321}, Q_{3124,4231}, Q_{2314,3421}, Q_{1324,3241}, Q_{1324,2431}, Q_{1234,3142}$	
$(-3, -3, 3, 3; 20, -10, -10, -10, -10, 20; 3, 3, -3, -3)$	$Q_{2413,4321}, Q_{1423,4231}, Q_{1342,4231}, Q_{1324,4132}, Q_{1324,4213}, Q_{1234,3142}$	
$(3, -7, -7, 11; 2, 2, 4, 4, -6, -6; 3, -7, -7, 11)$	$Q_{3142,4321}, Q_{3124,4312}, Q_{1342,3421}, Q_{2134,3412}, Q_{1243,3412}, Q_{1234,2413}$	
$(11, -1, -7, -3; -2, -8, -4, -4, 0, 18; 11, -1, -7, -3)$	$Q_{2413,4321}, Q_{2143,4231}, Q_{2134,4213}, Q_{1243,2431}, Q_{1234,2413}$	(24, 45, 27, 5)
$(-3, -7, -1, 11; 18, 0, -4, -4, -8, -2; -3, -7, -1, 11)$	$Q_{3142,4321}, Q_{3124,4312}, Q_{1342,3421}$ $Q_{1324,3412}, Q_{1234,3142}$	

Subdivisions of Perm_4 into BIPs using $\text{Tr}\,\text{Fl}_4^{>0}$



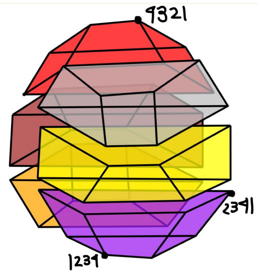
- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\text{Tr}\,\text{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\text{Tr}\,\text{Fl}_n^{>0}$ more generally.
  How many maximal cones?

Subdivisions of Perm_4 into BIPs using $\mathrm{Tr}\,\mathrm{Fl}_4^{>0}$

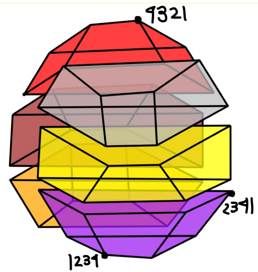


- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\mathrm{Tr}\,\mathrm{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ more generally.
  How many maximal cones?

Subdivisions of Perm_4 into BIPs using $\text{Tr}\,\text{Fl}_4^{>0}$

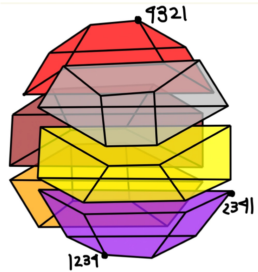


- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\text{Tr}\,\text{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\text{Tr}\,\text{Fl}_n^{>0}$ more generally. How many maximal cones?



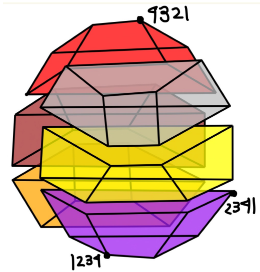
- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\text{Tr}\,\text{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\text{Tr}\,\text{Fl}_n^{>0}$ more generally. How many maximal cones?

Subdivisions of Perm_4 into BIPs using $\text{Tr}\,\text{Fl}_4^{>0}$



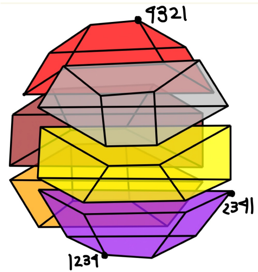
- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\text{Tr}\,\text{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\text{Tr}\,\text{Fl}_n^{>0}$ more generally. How many maximal cones?

Subdivisions of Perm_4 into BIPs using $\mathrm{Tr}\,\mathrm{Fl}_4^{>0}$



- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\mathrm{Tr}\,\mathrm{Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\mathrm{Tr}\,\mathrm{Fl}_n^{>0}$ more generally.
  How many maximal cones?

Subdivisions of Perm_4 into BIPs using $\mathrm{Tr\,Fl}_4^{>0}$



- The first subdivision in previous table is the one from Nadeau-Tewari.
- $\mathrm{Tr\,Fl}_4^{>0}$ is a polyhedral fan with 14 cones.
  Has same combinatorics as associahedron, and is closely connected to cluster algebra of type A_3 (which is the cluster type of Fl_4).
- Open question: describe combinatorics of $\mathrm{Tr\,Fl}_n^{>0}$ more generally.
  How many maximal cones?

Next: we will generalize the previous theorem in two ways:

- The previous theorem was for about the (complete) flag variety. We will extend to more general partial flag varieties.
- We will replace the adjectives "positive" by "nonnegative", which allows us to look at subdivisions of more general polytopes (not just permutohedron).

Polyhedral and tropical geometry of flag positroids

Next: we will generalize the previous theorem in two ways:

- The previous theorem was for about the (complete) flag variety. We will extend to more general partial flag varieties.
- We will replace the adjectives “positive” by “nonnegative”, which allows us to look at subdivisions of more general polytopes (not just permutohedron).

Polyhedral and tropical geometry of flag positroids

Next: we will generalize the previous theorem in two ways:

- The previous theorem was for about the (complete) flag variety. We will extend to more general partial flag varieties.
- We will replace the adjectives "positive" by "nonnegative", which allows us to look at subdivisions of more general polytopes (not just permutohedron).

Polyhedral and tropical geometry of flag positroids

Next: we will generalize the previous theorem in two ways:

- The previous theorem was for about the (complete) flag variety. We will extend to more general partial flag varieties.
- We will replace the adjectives "positive" by "nonnegative", which allows us to look at subdivisions of more general polytopes (not just permutohedron).

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $Fl_{R;n}$ is

$$Fl_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $Gr_{r,n}$.

Can represent an element of $Fl_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $\mathrm{Fl}_{R;n}$ is

$$\mathrm{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $\mathrm{Gr}_{r,n}$.

Can represent an element of $\mathrm{Fl}_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $\mathrm{Fl}_{R;n}$ is

$$\mathrm{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $\mathrm{Gr}_{r,n}$.

Can represent an element of $\mathrm{Fl}_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $\mathrm{Fl}_{R;n}$ is

$$\mathrm{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $\mathrm{Gr}_{r,n}$.

Can represent an element of $\mathrm{Fl}_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $Fl_{R;n}$ is

$$Fl_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $Gr_{r,n}$.

Can represent an element of $Fl_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $\mathrm{Fl}_{R;n}$ is

$$\mathrm{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $\mathrm{Gr}_{r,n}$.

Can represent an element of $\mathrm{Fl}_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

The flag variety

Let $R = \{r_1 < \cdots < r_k\} \subseteq [n] = \{1, 2, \ldots, n\}$. The *flag variety* $\mathrm{Fl}_{R;n}$ is

$$\mathrm{Fl}_{R;n} = \{(V_1, \ldots, V_k) \mid 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{R}^n \text{ and } \dim V_i = r_i \ \forall i\}$$

Special cases:

- If $R = [n]$: complete flag variety Fl_n
- If $R = \{r\}$: Grassmannian $\mathrm{Gr}_{r,n}$.

Can represent an element of $\mathrm{Fl}_{R;n}$ by an $r_k \times n$ matrix A such that the span of the top r_i rows is V_i.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ represents an element of Fl_3.

For $I \subset [n]$, the *Plücker coordinate* (or *flag minor*) $p_I(A)$ is the determinant of the submatrix in columns I and rows $1, 2, \ldots, |I|$.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope.*

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope.*

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope.*

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \dots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \dots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \dots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Flag matroids, flag positroids, and their polytopes

Let $R = (r_1 < \cdots < r_k) \subset [n]$. A *flag matroid* of ranks R is a sequence $\underline{M} = (M_1, \ldots, M_k)$ of matroids of ranks r_i on $[n]$ such that all vertices of

$$P(\underline{M}) = P(M_1) + \cdots + P(M_k) \quad \text{(Minkowski sum)}$$

are equidistant from the origin. $P(\underline{M})$ then called a *flag matroid polytope*.

If $\exists r_k \times n$ matrix A whose top r_i rows realize M_i $\forall i$, say A realizes $\underline{M}$.

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $R = (r_1 < \cdots < r_k) \subset [n]$.
Let A be a matrix giving a realization of flag matroid $\underline{M} = (M_1, \ldots, M_k)$.
If the flag minors $p_I(A) \geq 0$ for $|I| \in \{r_1, \ldots, r_k\}$, we say $\underline{M}$ is a *flag positroid* and $P(\underline{M})$ a *flag positroid polytope*.

Note: by definition, a flag positroid is realizable.

Polyhedral and tropical geometry of flag positroids

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \ldots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch-Karp).

Polyhedral and tropical geometry of flag positroids

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \dots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \dots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch–Karp).

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \ldots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch–Karp).

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \ldots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch–Karp).

Polyhedral and tropical geometry of flag positroids

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \dots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \dots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch-Karp).

Polyhedral and tropical geometry of flag positroids

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \ldots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$; that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch-Karp).

Polyhedral and tropical geometry of flag positroids

Theorem (Boretsky–Eur–W)

Let R be a sequence of consecutive integers $(a, a+1, \ldots, b) \subset [n]$. Let $\mu = (\mu^a, \mu^{a+1}, \ldots, \mu^b) \in \prod_{i=a}^{b} (\mathbb{R} \cup \{\infty\})^{\binom{[n]}{i}}$. TFAE:

- μ lies in the nonnegative tropical flag variety $\mathrm{Tr}\,\mathrm{Fl}_{R;n}^{\geq 0}$;
  that is, μ obeys all positive tropical 3-term Plücker relations and incidence Plücker relations.
- Every face in the coherent subdivision of the polytope $P(\underline{\mu^a}) + \cdots + P(\underline{\mu^b})$ is a flag positroid polytope.

Note: we do not know if the result holds for arbitrary sequences $R = (r_1 < \cdots < r_k) \subset [n]$.[2]

[2] One ingredient in proof is fact that the two notions of positivity for flag varieties (Plücker positivity vs Lusztig positivity) coincide iff R a consecutive subset (Bloch-Karp).

Applications of main result

Corollary (Boretsky-Eur-W)

A complete flag matroid polytope is a Bruhat interval polytope iff all its 1 and 2-dimensional faces are.

Corollary (Boretsky-Eur-W)

More generally, given a flag matroid $\underline{M} = (M_a, M_{a+1}, \ldots, M_b)$ of consecutive ranks $a, a+1, \ldots, b$, its flag matroid polytope $P(\underline{M})$ is a flag positroid polytope if and only if all its 1 and 2-dimensional faces are.

Applications of main result

Corollary (Boretsky-Eur-W)

A complete flag matroid polytope is a Bruhat interval polytope iff all its 1 and 2-dimensional faces are.

Corollary (Boretsky-Eur-W)

More generally, given a flag matroid $\underline{M} = (M_a, M_{a+1}, \ldots, M_b)$ of consecutive ranks $a, a+1, \ldots, b$, its flag matroid polytope $P(\underline{M})$ is a flag positroid polytope if and only if all its 1 and 2-dimensional faces are.

Applications of main result

Corollary (Boretsky-Eur-W)

A complete flag matroid polytope is a Bruhat interval polytope iff all its 1 and 2-dimensional faces are.

Corollary (Boretsky-Eur-W)

More generally, given a flag matroid $\underline{M} = (M_a, M_{a+1}, \dots, M_b)$ of consecutive ranks $a, a+1, \dots, b$, its flag matroid polytope $P(\underline{M})$ is a flag positroid polytope if and only if all its 1 and 2-dimensional faces are.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \dots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \dots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \dots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \dots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \dots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \dots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \dots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \dots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is “consistent w/ 3-term Plücker relations.”
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."

Related notions of oriented flag matroid, positively oriented (flag) matroid.

Question: when does a sequence $(M_1, \dots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \dots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \dots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \dots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \dots, M_k)$ of positroids of ranks $r_1 < \dots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \dots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \dots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \dots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Applications of main result

Recall: an *oriented matroid* χ_M of rank d on $[n]$ is a map $\chi_M : \binom{[n]}{d} \to \{-1, 0, 1\}$ which is "consistent w/ 3-term Plücker relations."
Related notions of oriented flag matroid, positively oriented (flag) matroid.
Question: when does a sequence $(M_1, \ldots, M_k)$ of positroids of ranks $r_1 < \cdots < r_k$ have a realization by one matrix?

Corollary (Boretsky-Eur-W)

Suppose that $(M_a, \ldots, M_b)$ is a sequence of positroids of consecutive ranks. Then, when considered as a sequence of positively oriented matroids, $(M_a, \ldots, M_b)$ is a flag positroid iff it's an oriented flag matroid.

Corollary (Boretsky-Eur-W)

Every positively oriented flag matroid $(\chi_{M_1}, \ldots, \chi_{M_k})$ of consecutive ranks is realizable.

The above theorem generalizes a result of Ardila-Rincon-W.

Thank you for listening!

- "Polyhedral and tropical geometry of flag positroids," with Jon Boretsky and Chris Eur, arXiv:2208.09131.
- "Bruhat interval polytopes" with Emmanuel Tsukerman, Adv Math 2015.
- "The full Kostant-Toda hierarchy on the positive flag variety" with Yuji Kodama, Comm Math Phys 2015.

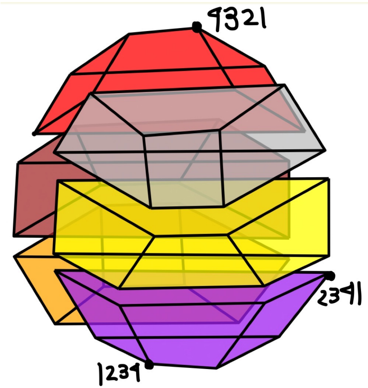


Figure from Nadeau-Tewari 2208.04128 "Remixed Eulerian numbers"

Quick intro to tropical geometry

- Tropical varieties are "piecewise-linear" versions of ordinary varieties.
- Let $\mathcal{C} = \cup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ be the field of *Puiseux series*. Each element has the form $ct^a + \dots$ where $c \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{Q}$, and the other terms have larger exponents.
- Define *valuation* $\mathrm{val}(ct^a + \dots) = a$.
- For $I \subset \mathcal{C}[x_1, \dots, x_n]$ an ideal, the corresponding *tropical variety* is $\mathrm{Trop}\, V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C} \setminus \{0\})^n)}$. That is, we compute the "ordinary" variety over the field of Puiseux series, but then apply the valuation to each coordinate.
- The *positive Puiseux series* $\mathcal{C}^+$ consists of $ct^a + \dots$ where $c \in \mathbb{R}_{>0}$.
- The *positive tropical variety* is $\mathrm{Trop}^+ V(I) := \overline{\mathrm{val}(V(I) \cap (\mathcal{C}^+ \setminus \{0\})^n)}$.
- Note: If $f(t), g(t) \in \mathcal{C}^+$, then $\mathrm{val}(f(t)g(t)) = \mathrm{val}(f(t) + g(t)$ and $\mathrm{val}(f(t) + g(t)) = \min(\mathrm{val}(f(t), \mathrm{val}(g(t)))$.

Polyhedral and tropical geometry of positroids

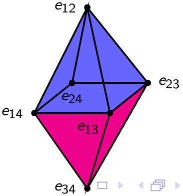
Thm (Lukowski-Parisi-W, Speyer-W, Arkani-Hamed-Lam-Spradlin)

Let $\mu = (\mu_I \mid I \in \binom{[n]}{r}) \in \mathbb{R}^{\binom{[n]}{r}}$. The following are equivalent.

- μ lies in the positive tropical Grassmannian $\mathrm{Tr}\,\mathrm{Gr}_{r,n}^{>0}$ (the closure of the coordinate-wise valuation of $\mathrm{Gr}_{r,n}$ over positive Puiseux series)
- μ obeys the positive tropical 3-term Plücker relations: for $i < j < k < \ell$ and S disjoint from them, $|S| = r - 2$,
  $$\mu_{Sik} + \mu_{Sj\ell} = \min(\mu_{Sij} + \mu_{Sk\ell}, \mu_{Si\ell} + \mu_{Sjk}).$$
- Every face in the regular (coherent) subdivision of the hypersimplex $\Delta_{r,n} = \mathrm{Conv}(e_I \mid I \in \binom{[n]}{r}))$ induced by μ is a positroid polytope.

Example: $n = 4, r = 2, (\mu_I) \in \mathbb{R}^{\binom{[4]}{2}}$ such that

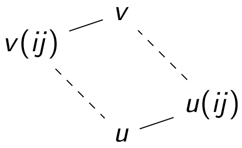
$\mu_{13} + \mu_{24} = \mu_{23} + \mu_{14} < \mu_{12} + \mu_{34}$. Get:



.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



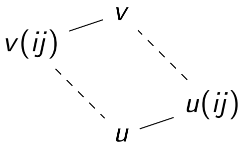
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



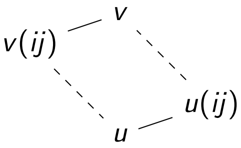
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



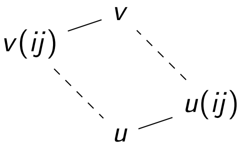
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

Example of the Generalized lifting property

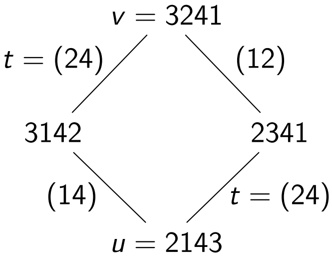
Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in the Bruhat order on S_n. Choose a transposition $t = (ij)$ which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

Example of the Generalized lifting property

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in the Bruhat order on S_n. Choose a transposition $t = (ij)$ which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.



Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad \text{if } u \not\leq v.$
2. $R_{u,v}(q) = 1, \quad \text{if } u = v.$
3. If $vs < v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us < u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us > u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad \text{if } u \not\leq v.$
2. $R_{u,v}(q) = 1, \quad \text{if } u = v.$
3. If $vs \lessdot v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation:
$R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad$ if $u \not\leq v$.
2. $R_{u,v}(q) = 1, \quad$ if $u = v$.
3. If $vs \lessdot v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad \text{if } u \not\leq v.$
2. $R_{u,v}(q) = 1, \quad \text{if } u = v.$
3. If $vs \lessdot v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.