Baxter $d$-permutations and other pattern avoiding classes

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Permutations and diagrams

A permutation $\sigma = \sigma(1), \ldots, \sigma(n) \in S_n$ is a bijection from $[n] := \{1, 2, \ldots, n\}$ to itself.
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Example: the diagram of $\sigma = 413526$
Permutations and diagrams

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Example: the diagram of $\sigma = 413526$

A diagram (of size $n$) is a point set on $[n] \times [n]$ with exactly 1 point per row and 1 point per column. $S_n$ the set of permutations of size $n$. $|S_n| = n!$
The Starting Question

- Permutations ⇔ Diagrams
- Diagrams are 2D objects
- What could be a ”3D” diagram?
- ⇒ ”3D” Permutations?
A 3-diagram of size $n$ is a point set on $[n]^3$ such that
**d-Diagrams and d-Permutations**

A **3-diagram** of size $n$ is a point set on $[n]^3$ such that each plane of the grid (orthogonal to $x, y$ or $z$) contains exactly 1 point.

Points:
- $(1,2,6)$
- $(2,5,5)$
- $(3,3,4)$
- $(4,1,3)$
- $(5,4,2)$
- $(6,6,1)$

A **$d$-diagram** is a point set of size $n$ on $[n]^d$ such that each **hyperplane** orthogonal $x_i = j$ with $i \in [d]$ and $j \in [n]$ contains exactly 1 point.
A **3-permutation** $\sigma := (\sigma_y, \sigma_z)$ is a pair of permutations. $S_n^2 := \{3\text{-permutations of size } n\}$. $|S_n^2| = n!^2$.

A **$d$-permutation** of size $n$, $\sigma := (\sigma_1, \ldots, \sigma_{d-1})$ is a sequence of $d - 1$ permutations of size $n$. $S_n^{d-1} := \{d\text{-permutations of size } n\}$. $|S_n^{d-1}| = n!^{d-1}$.
Projections

Let $\overline{\sigma} := (\text{Id}_n, \sigma_1, \ldots, \sigma_{d-1})$. The projection on $i$ of $d$-permutation $\sigma$ is the $d'$-permutation $\text{proj}_i(\sigma) := \overline{\sigma}_{i_2} \overline{\sigma}_{i_3}^{-1}, \ldots, \overline{\sigma}_{i_{d'}} \overline{\sigma}_{i_1}^{-1}$. $d'$ is the dimension of the projection.
Projections

\[
\text{proj}_{xy}(\sigma) = \sigma_y,
\]
\[
\text{proj}_{xz}(\sigma) = \sigma_z,
\]
\[
\text{proj}_{yz}(\sigma) = \sigma_z \sigma_y^{-1}.
\]
\[
\text{proj}_{yz}((253146, 654321)) = 364251.
\]
\[
\text{proj}_{y,x}(\sigma) = \sigma_y^{-1}.
\]

\[\mathbf{i} := i_1, \ldots, i_{d'} \in [d]^{d'}, \text{ the projection } \text{proj}_i \text{ is } \text{direct} \text{ if } i_1 < i_2 < \cdots < i_{d'}.
\]

Let \(\overline{\sigma} := (\text{Id}_n, \sigma_1, \ldots, \sigma_{d-1})\), the projection on \(i\) of \(d\)-permutation \(\sigma\) is the \(d'\)-permutation \(\text{proj}_i(\sigma) := \sigma_{i_2} \overline{\sigma}_{i_1}^{-1}, \sigma_{i_3} \overline{\sigma}_{i_1}^{-1}, \ldots, \sigma_{i_{d'}} \overline{\sigma}_{i_1}^{-1}\). \(d'\) is the dimension of the projection.
Pattern (classic)

A permutation $\sigma$ contains a permutation (or a pattern) $\pi = \pi(1), \ldots, \pi(k) \in S_k$ if there exist indices $c_1 < \cdots < c_k$ such that $\sigma(c_1) \cdots \sigma(c_k)$ is order-isomorphic to $\pi$.

The set of points of indices $c_1, \cdots, c_k$ is an occurrence of the $\pi$.

$\sigma = 413526$ contains [several occurrences of] the pattern $\pi = 213$. 

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    axis equal image,
    xlabel={x},
    ylabel={y},
    grid=both,
    grid style={line width=0.5pt, draw=gray!60},
    enlargelimits=false,
    axis lines=middle,
]
\addplot[only marks, mark options={scale=0.5}, mark=*, mark size=3pt] table {x y}
\end{axis}
\end{tikzpicture}
\end{center}
Pattern (classic)

A permutation $\sigma$ contains a permutation (or a pattern) $\pi = \pi(1), \ldots, \pi(k) \in S_k$ if there exist indices $c_1 < \cdots < c_k$ such that $\sigma(c_1) \cdots \sigma(c_k)$ is order-isomorphic to $\pi$.

The set of points of indices $c_1, \cdots, c_k$ is an occurrence of the $\pi$.

$\sigma = 413526$ contains [several occurrences of] the pattern $\pi = 213$.

$$S_n(\pi) := \text{the set of permutations that avoids } \pi.$$  
$$S_n(\pi_1, \ldots, \pi_k) := \ldots \text{ avoids all } \pi_1, \ldots, \pi_k.$$
Pattern avoidance classes

\[ S_n(21) = \{Id_n\} \]
\[ S_n(21) = \{ Id_n \} \]

[Knuth 73]: \[ |S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots \]
Pattern avoidance classes

\[ S_n(21) = \{Id_n\} \]

[Knuth 73]: \[ |S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots \]

\[ \pi \text{ and } \tau \text{ are Wilf-equivalent of } |S_n(\pi)| = |S_n(\tau)|. \]

\[ |S_n(21)| = |S_n(12)| = 1 \]
Pattern avoidance classes

$S_n(21) = \{Id_n\}$

[Knuth 73]: $|S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots$

$\pi$ and $\tau$ are **Wilf-equivalent** of $|S_n(\pi)| = |S_n(\tau)|$.

$|S_n(21)| = |S_n(12)| = 1$

$\pi$ and $\tau$ are **trivially Wilf-equivalent** if there is a symmetry $s$ of the square such that $\forall \sigma, \sigma \in S(\pi)$ iff $s(\sigma) \in S(s(\tau))$. 
# Pattern avoidance classes

<table>
<thead>
<tr>
<th>Patterns</th>
<th>(1)</th>
<th>Sequence</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>2</td>
<td>1, 1, 1, 1, 1, 1, 1, 1, ...</td>
<td></td>
</tr>
<tr>
<td>12, 21</td>
<td>1</td>
<td>1, 0, 0, 0, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>312</td>
<td>4</td>
<td>$\frac{1}{n+1}\binom{2n}{n} = 1, 2, 5, 14, 42, 132, ...$</td>
<td>[Knuth 73]</td>
</tr>
<tr>
<td>123</td>
<td>2</td>
<td>$\frac{1}{n+1}\binom{2n}{n} = 1, 2, 5, 14, 42, 132, ...$</td>
<td>[Knuth 73]</td>
</tr>
<tr>
<td>123, 321</td>
<td>1</td>
<td>1, 2, 4, 4, 0, 0, 0, ...</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>213, 321</td>
<td>4</td>
<td>$1 + \frac{n(n-1)}{2} = 1, 2, 4, 7, 11, 16, 22, ...$</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>312, 231</td>
<td>2</td>
<td>$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>231, 132</td>
<td>4</td>
<td>$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>312, 321</td>
<td>4</td>
<td>$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>213, 132, 123</td>
<td>2</td>
<td>Fibonacci: 1, 2, 3, 5, 8, 13, 21, ...</td>
<td>[Simion 85]</td>
</tr>
<tr>
<td>231, 213, 321</td>
<td>8</td>
<td>$n = 1, 2, 3, 4, 5, 6, 7, ...$</td>
<td>[Simion 85]</td>
</tr>
<tr>
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<td>4</td>
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</tr>
<tr>
<td>312, 321, 123</td>
<td>4</td>
<td>1, 2, 3, 1, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>321, 213, 123</td>
<td>4</td>
<td>1, 2, 3, 1, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>321, 213, 132</td>
<td>2</td>
<td>$n = 1, 2, 3, 4, 5, 6, 7, ...$</td>
<td>[Simion 85]</td>
</tr>
</tbody>
</table>

(1): Number of trivially Wilf-Equivalent patterns.
patterns and \(d\)-permutations

\[\sigma \text{ contains a pattern } \pi \text{ if}\]

\[\text{proj}_{xy}, \text{proj}_{xz}, \text{or proj}_{yz}\text{ contains } \pi \text{ if}\]

\[\text{(1432, 3124) contains the pattern (132, 213).}\]
patterns and $d$-permutations

$\sigma$ contains a pattern $\pi$ if there exists a subset of points of $\sigma$ that is equal (once standardized) to $\pi$.

$(1432, 3124)$ contains the pattern $(132, 213)$. 
patterns and $d$-permutations

$\sigma$ contains a pattern $\pi$ if there exists a subset of points of $\sigma$ that is equal (once standardized) to $\pi$.

$$(1432, 3124)$$ contains the pattern $(132, 213)$.

$\sigma$ contains a pattern $\pi_1$ if $\text{proj}_{xy}, \text{proj}_{xz}$ or $\text{proj}_{yz}$ contains $\pi_1$. 
patterns and $d$-permutations

Let $\sigma \in S_{n}^{d-1}$ and $\pi \in S_{k}^{d'-1}$ with $k \leq n$. Then $\sigma$ contains the pattern $\pi$, if there exist a direct projection $\sigma' = \text{proj}_i(\sigma)$ of dimension $d'$ that contains $\pi$.

(1432, 3124) contains the pattern (132, 213) and the pattern 231.

$\sigma$ contains the pattern $\pi$ if there are indices $c_1 < \cdots < c_k$ such that $\sigma'_i(c_1) \cdots \sigma'_i(c_k)$ is order-isomorphic to $\pi_i$ for all $i \in [d']$. 
patterns and $d$-permutations

Let $\sigma \in S_{n}^{d-1}$ and $\pi \in S_{k}^{d'-1}$ with $k \leq n$. Then $\sigma$ contains the pattern $\pi$, if there exist a direct projection $\sigma' = \text{proj}_i(\sigma)$ of dimension $d'$ that contains $\pi$.

$(1432, 3124)$ contains the pattern $(132, 213)$ and the pattern 231. Remark 1: $(132, 312)$ doesn’t contain $(12, 12)$ but 132 and 312 both contain the pattern 12 (but on different positions).

$\sigma$ contains the pattern $\pi$ if there are indices $c_1 < \cdots < c_k$ such that $\sigma'_i(c_1) \cdots \sigma'_i(c_k)$ is order-isomorphic to $\pi_i$ for all $i \in [d']$. 
3-Pattern avoidance classes

<table>
<thead>
<tr>
<th>Patterns</th>
<th>(1)</th>
<th>Sequence</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12, 12)</td>
<td>4</td>
<td>1, 3, 17, 151, 1899, 31711, ⋯</td>
<td>weak-Bruhat interval</td>
</tr>
<tr>
<td>(12, 12), (12, 21)</td>
<td>6</td>
<td>$n! = 1, 2, 6, 24, 120 ⋯$</td>
<td>$\sigma_1 \Rightarrow \sigma_2$</td>
</tr>
<tr>
<td>(12, 12), (12, 21), (21, 12)</td>
<td>4</td>
<td>1, 1, 1, 1, 1, 1, ⋯</td>
<td>1 diagonal</td>
</tr>
<tr>
<td>(12, 12), (12, 21), (21, 12), (21, 21)</td>
<td>1</td>
<td>1, 0, 0, 0, 0, 0, ⋯</td>
<td></td>
</tr>
<tr>
<td>(123, 123)</td>
<td>4</td>
<td>1, 4, 35, 524, 11774, 366352, 14953983, ⋯</td>
<td>new</td>
</tr>
<tr>
<td>(123, 132)</td>
<td>24</td>
<td>1, 4, 35, 524, 11768, 365558, 14871439, ⋯</td>
<td>new</td>
</tr>
<tr>
<td>(132, 213)</td>
<td>8</td>
<td>1, 4, 35, 524, 11759, 364372, 14748525, ⋯</td>
<td>new</td>
</tr>
<tr>
<td>(12, 12), (132, 312)</td>
<td>48</td>
<td>$(n + 1)^{n-1} = 1, 3, 16, 125, 1296 ⋯$</td>
<td>[Atkinson et al. 93]</td>
</tr>
<tr>
<td>(12, 12), (123, 321)</td>
<td>12</td>
<td>1, 3, 16, 124, 1262, 15898, ⋯</td>
<td>distributive lattice</td>
</tr>
<tr>
<td>(12, 12), (231, 312)</td>
<td>8</td>
<td>1, 3, 16, 122, 1188, 13844, ⋯</td>
<td>A295928?</td>
</tr>
</tbody>
</table>

(1): Number of trivially Wilf-Equivalent patterns.
### 2-Pattern avoidance classes

<table>
<thead>
<tr>
<th>Patterns</th>
<th>(1)</th>
<th>Sequence</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1</td>
<td>1, 0, 0, 0, 0, 0, ...</td>
<td>unavoidable pattern</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>1, 1, 1, 1, 1, ...</td>
<td>1 diagonal</td>
</tr>
<tr>
<td>123</td>
<td>1</td>
<td>1, 4, 20, 100, 410, 1224, 2232, ...</td>
<td>new</td>
</tr>
<tr>
<td>132</td>
<td>2</td>
<td>1, 4, 21, 116, 646, 3596, 19981, ...</td>
<td>new</td>
</tr>
<tr>
<td>231</td>
<td>2</td>
<td>1, 4, 21, 123, 767, 4994, 33584, ...</td>
<td>new</td>
</tr>
<tr>
<td>321</td>
<td>1</td>
<td>1, 4, 21, 128, 850, 5956, 43235, ...</td>
<td>new</td>
</tr>
<tr>
<td>123, 132</td>
<td>2</td>
<td>1, 4, 8, 8, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>123, 231</td>
<td>2</td>
<td>1, 4, 9, 6, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>123, 321</td>
<td>1</td>
<td>1, 4, 8, 0, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>132, 213</td>
<td>1</td>
<td>1, 4, 12, 28, 58, 114, 220, ...</td>
<td>[Sun22+]</td>
</tr>
<tr>
<td>132, 231</td>
<td>4</td>
<td>1, 4, 12, 32, 80, 192, 448, ...</td>
<td>A001787 [Sun22+]</td>
</tr>
<tr>
<td>132, 321</td>
<td>2</td>
<td>1, 4, 12, 27, 51, 86, 134, ...</td>
<td>A047732 [Sun22+]</td>
</tr>
<tr>
<td>231, 312</td>
<td>1</td>
<td>1, 4, 10, 28, 76, 208, 568, ...</td>
<td>A026150 [Sun22+]</td>
</tr>
<tr>
<td>231, 321</td>
<td>2</td>
<td>1, 4, 12, 36, 108, 324, 972, ...</td>
<td>A003946 [Sun22+]</td>
</tr>
</tbody>
</table>

(1): Number of trivially Wilf-Equivalent patterns.
1— and 2-Patterns avoidance classes

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<th>Patterns</th>
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</tr>
</thead>
<tbody>
<tr>
<td>12, (12, 12)</td>
<td>1</td>
<td>1, 0, 0, 0, 0, 0, ...</td>
<td>12</td>
</tr>
<tr>
<td>12, (21, 12)</td>
<td>3</td>
<td>1, 0, 0, 0, 0, 0, ...</td>
<td>12</td>
</tr>
<tr>
<td>21, (12, 12)</td>
<td>1</td>
<td>1, 0, 0, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>21, (21, 12)</td>
<td>3</td>
<td>1, 1, 1, 1, 1, 1, ...</td>
<td>21</td>
</tr>
<tr>
<td>123, (12, 12)</td>
<td>1</td>
<td>1, 3, 14, 70, 288, 822, 1260, ...</td>
<td>new</td>
</tr>
<tr>
<td>123, (12, 21)</td>
<td>3</td>
<td>1, 3, 6, 6, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>132, (12, 12)</td>
<td>2</td>
<td>1, 3, 11, 41, 153, 573, 2157, ...</td>
<td>A0281593?</td>
</tr>
<tr>
<td>132, (12, 21)</td>
<td>6</td>
<td>1, 3, 11, 43, 173, 707, 2917, ...</td>
<td>A026671?</td>
</tr>
<tr>
<td>231, (12, 12)</td>
<td>2</td>
<td>1, 3, 9, 26, 72, 192, 496, ...</td>
<td>A072863?</td>
</tr>
<tr>
<td>231, (12, 21)</td>
<td>4</td>
<td>1, 3, 11, 44, 186, 818, 3706, ...</td>
<td>new</td>
</tr>
<tr>
<td>231, (21, 12)</td>
<td>2</td>
<td>1, 3, 12, 55, 273, 1428, 7752, ...</td>
<td>A001764?</td>
</tr>
<tr>
<td>321, (12, 12)</td>
<td>1</td>
<td>1, 3, 2, 0, 0, 0, 0, ...</td>
<td></td>
</tr>
<tr>
<td>321, (12, 21)</td>
<td>3</td>
<td>1, 3, 11, 47, 221, 1113, 5903, ...</td>
<td>A217216?</td>
</tr>
</tbody>
</table>

(1): Number of trivially Wilf-Equivalent patterns.
Separable permutations $Sep_n$

**direct sum** $\sigma \oplus \pi$: add $\pi$ in the top right corner of $\sigma$.

**skew sum** $\sigma \ominus \pi$: add $\pi$ in the bottom right corner of $\sigma$.

**separable**: size 1 or a direct/skew sum separable permutations.

---

$\sigma$ and $\pi$ two permutations respectively of size $n$ and $k$.

$\sigma \oplus \pi := \sigma(1), \ldots, \sigma(n), \pi(1) + k, \ldots, \pi(k) + n$ and

$\sigma \ominus \pi := \sigma(1) + k, \ldots, \sigma(n) + k, \pi(1), \ldots, \pi(k)$. 
Separable permutations $Sep_n$

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**separable**: size 1 or a direct/skew sum separable permutations.

On the left the separable permutation $643512 = 1 \ominus ((1 \ominus 1) \oplus 1) \ominus (1 \oplus 1)$. 

---

$| Sep_n | = n^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k} + 2^{n-k-1}$

---

$Sep_n = S_n(2413, 3142)$

---

[Bose Buss Lubiw 98]
Separable permutations $Sep_n$

**direct sum** $\sigma \oplus \pi$: add $\pi$ in the top right corner of $\sigma$.

**skew sum** $\sigma \ominus \pi$: add $\pi$ in the bottom right corner of $\sigma$.

**separable**: size 1 or a direct/skew sum separable permutations.

$|Sep_n| = n - 1 \cdot \sum_{k=0}^{n-2} \left( n - 1 \right)^k + 2^{n-k-1}$

$Sep_n = S_n(2413, 3142)$

[Bose Buss Lubiw 98]
Separable permutations $Sep_n$

**direct sum** $\sigma \oplus \pi$: add $\pi$ in the top right corner of $\sigma$.

**skew sum** $\sigma \ominus \pi$: add $\pi$ in the bottom right corner of $\sigma$.

**separable**: size 1 or a direct/skew sum separable permutations.

[Brightwell 92]:

$$|Sep_n| = \frac{1}{n - 1} \sum_{k=0}^{n-2} \binom{n - 1}{k} \binom{n - 1}{k + 1} 2^{n-k-1}.$$
Separable permutations $Sep_n$

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[Brightwell 92]:

$$|Sep_n| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} 2^{n-k-1}.$$ 

$Sep_n = S_n(2413, 3142)$ [Bose Buss Lubiw 98]
Separable \( d \)-permutations \([\text{Atkinson Mansour 10}]\)

**separable \( d \)-permutation:** size 1 or a \( d \)-sum separable permutations.

Let \( \sigma \) and \( \pi \) two \( d \)-permutations and \( \text{dir} \in \{+, -\}^d \). The \( d \)-**sum** with respect to direction \( \text{dir} \) is:

\[
\sigma \oplus^\text{dir} \pi := \overline{\sigma}_2 \oplus^\text{dir}_2 \overline{\pi}_2, \ldots, \overline{\sigma}_d \oplus^\text{dir}_d \overline{\pi}_d,
\]

where \( \oplus^\text{dir}_i \) is \( \oplus \) if \( \text{dir}_i = + \) and \( \ominus \) if \( \text{dir}_i = - \).

\[
p_1 = (132, 132) = (1, 1) \oplus^{(+++)} ((1, 1) \oplus^{(+-)} (1, 1))
\]
Separable $d$-permutations [Atkinson Mansour 10]
Separable $d$-permutations

\[ Sep_{n}^{d-1} = S_{n}^{d-1}(\text{Sym}((132, 213)), 2413, 3142) \]
Separable $d$-permutations

\[ \text{Sep}_{n}^{d-1} = S_{n}^{d-1}(\text{Sym}((132, 213)), 2413, 3142) \]

\[ |\text{Sep}_{n}^{d-1}| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} (2^{d-1} - 1)^k (2^{d-1})^{n-k-1}. \]
Vincular Patterns and Baxter permutations

**vincular pattern**: a pattern where some entries must be consecutive in the permutation (adjacencies). Ex: $2413|_2$ and $3142|_2$
**Vincular Patterns and Baxter permutations**

**vincular pattern**: a pattern where some entries must be consecutive in the permutation (adjacencies). Ex: $2413\uparrow_2$ and $3142\uparrow_2$
Vincular Patterns and Baxter permutations

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Baxter permutations: $B_n := S_n(2413|_2, 3142|_2)$. 
**Generalized vincular Patterns**

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Natural extension of generalized vincular patterns to $d$-permuations...
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Natural extension of generalized vincular patterns to $d$-permutations...

But what could be a Baxter $d$-permutation...
Well sliced permutations

A **slice** is a rectangle defined by two adjacent points. **type:** horizontal or vertical. The **direction** of a slice is: $+$ or $-$. 
Well sliced permutations

A slice is a rectangle defined by two adjacent points. type: horizontal or vertical. The direction of a slice is: + or -.

well-sliced: each slice intersects exactly 1 slice of each type and two intersecting slices share the same direction.
Well sliced permutations

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Proposition: Baxter ≡ well-sliced
Well-sliced $d$-permutations

The **direction** of a slice is: $++, +-, -+, --$. The **type** of a slice is: $x, y$ or $z$.

A **Baxter $d$-permutation** is a $d$-permutation such that each of its $d' \leq d$ projection is well-sliced.
Well-sliced $d$-permutations vs Baxter $d$-permutation

well-sliced but not Baxter.
Baxter $d$-permutation

well-sliced but not Baxter.
Baxter $d$-permutation characterisation

\[ B_{n}^{d-1} = S_{n}^{d-1}(\text{Sym}(2413|_{2,2}), \text{Sym}((312, 213)|_{1,2,.}), \text{Sym}((3412, 1432)|_{2,2,.}), \text{Sym}((2143, 1423)|_{2,2,.})). \]
Baxter $d$-permutation enumeration

| $|B^d_{n-1}|$ |
|---|
| $n/d$ | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 4 | 8 | 16 |
| 3 | 6 | 28 | 120 | 496 |
| 4 | 22 | 260 | 2440 | 20816 |
| 5 | 92 | 2872 | 59312 | 1035616 |
| 6 | 422 | 35620 |  |  |
| 7 | 2074 | 479508 |  |  |
[Bonichon Bousquet Fusy 10] There is a bijection between Baxter permutations and plane bipolar orientations.
Maps ?

([2, 5, 3, 1, 4, 6], [6, 5, 4, 3, 2, 1])
conclusion/perspectives

- nice framework
- nice generalisation of Baxter permutation
- lot of open problems.
- implementation available
  plmlab.math.cnrs.fr/bonichon/multipermutation

Have fun!
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