Calcul de caractères d'algèbres de Lie avec les partitions d'entiers

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Outline

Lie algebras

Definition

A *Lie algebra* $\mathfrak{g}$ is a vector space together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying:

- alternativity : for all $x \in \mathfrak{g}$, $[x, x] = 0$,
- the Jacobi identity: for all $x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

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Example

The *special linear Lie algebra* of order n, denoted A_{n-1} or $\mathfrak{sl}_n(\mathbb{C})$, is the Lie algebra of $n \times n$ matrices with trace zero and with the Lie bracket $[X, Y] = XY - YX$.

Representations

Definition

A *representation (or module)* of $\mathfrak{g}$ is a vector space V together with a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, such that

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

By abuse of notation, V is often called a $\mathfrak{g}$-module and $\rho(X)(v)$ is often written $X \cdot v$.

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Examples

- trivial representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho(X) = 0$ for all $X \in \mathfrak{g}$,
- adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $ad(X)(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

Semi-simple Lie algebras

Definition

Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if

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Definition

A Lie algebra $\mathfrak{g}$ is said to be *semi-simple* if it is a direct sum of simple Lie algebras.

Semi-simple Lie algebras can be described in terms of generators and relations.

Infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra.
It is possible to define and *affine Kac-Moody Lie algebra* $\hat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ as

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t, and $\mathbb{C}c$ is $\hat{\mathfrak{g}}$'s center (one-dimensional) which satisfies $[c, g] = 0$ for all $g \in \mathfrak{g}$.

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Kac-Moody Lie algebras can also be described in terms of generators and relations.

Weights

Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ (nilpotent subalgebra which is self-normalizing, i.e. if $\forall X \in \mathfrak{h}, [X, Y] \in \mathfrak{h}$, then $Y \in \mathfrak{h}$).

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Definition

Let V be a $\mathfrak{g}$-module and μ be a linear functional on $\mathfrak{h}$. The *weight space* of V with weight μ is $V_\mu := \{v \in V : \forall H \in \mathfrak{h}, \quad H \cdot v = \mu(H)v\}$. A *weight* is a linear functional μ such that V_μ is non-zero.
If V is a direct sum $V = \bigoplus_\mu V_\mu$ of its weight spaces, then it is called a *weight module*.

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The *roots* are weights for the adjoint representation.
A weight λ is *higher* than another weight μ if $\lambda - \mu$ can be written as a sum of positive roots, and λ is a *highest weight* if it is higher than any other weight in V.

Outline

Characters

Definition

Let $L(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ be an irreducible highest weight module with highest weight λ. The *character* $\mathrm{ch} L(\lambda)$ of V is defined as

$$\mathrm{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu) e^\mu,$$

where e^μ is a formal exponential satisfying $e^\mu e^{\mu'} = e^{\mu+\mu'}$.

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where e^μ is a formal exponential satisfying $e^\mu e^{\mu'} = e^{\mu+\mu'}$.

By definition of a highest weight,

$$e^{-\lambda} \mathrm{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} \dim(V_\mu) e^{\mu-\lambda}$$

is a **series with positive coefficients** in $\mathbb{Z}[[e^{-\alpha_0}, \ldots, e^{-\alpha_n}]]$, where $\alpha_0, \ldots, \alpha_n$ are the simple roots.

Character formulas

Theorem (Weyl–Kac character formula)

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{dim\mathfrak{g}_\alpha}},$$

where W is the Weyl group of $\mathfrak{g}$, Δ^+ the set of positive roots of $\mathfrak{g}$, $\mathrm{sgn}(w)$ the signature of w, $\rho \in \mathfrak{h}^*$ the Weyl vector, and $\mathfrak{g}_\alpha$ the α root space of $\mathfrak{g}$.

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Principal specialisation ($e^{-\alpha_i} \mapsto q$ for all i) gives an infinite product.

Example: $A_1^{(1)}$ at level 3 (Lepowsky–Wilson)

$$e^{-\Lambda_0+2\Lambda_1} \mathrm{ch} L(\Lambda_0 + 2\Lambda_1) = \frac{(-q;q)_\infty}{(q,q^4;q^5)_\infty}, e^{-3\Lambda_1} \mathrm{ch} L(3\Lambda_1) = \frac{(-q;q)_\infty}{(q^2,q^3;q^5)_\infty},$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ and $(a,b;q)_n = (a;q)_n(b;q)_n$.

Digression: The Rogers–Ramanujan identities

Definition

A *partition* λ of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition λ.

Example

There are 5 partitions of 4: $4, (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$.

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- The generating function for partitions into distinct parts congruent to $k \mod N$ is
$$(-zq^k; q^N)_\infty.$$
- The generating function for partitions into parts congruent to $k \mod N$ is
$$\frac{1}{(zq^k; q^N)_\infty}.$$

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Theorem (Rogers 1894, Rogers–Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$(-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = (-q;q)_\infty \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

Obtained by giving two different formulations for the principal specialisation

$$e^{-\alpha_0} \mapsto q, \quad e^{-\alpha_1} \mapsto q$$

of $e^{-\Lambda_0+2\Lambda_1}\mathrm{ch}L(\Lambda_0+2\Lambda_1)$, where $L(\Lambda_0+2\Lambda_1)$ is an irreducible highest weight $A_1^{(1)}$-module of level 3 with highest weight $\Lambda_0+2\Lambda_1$, and α_0, α_1 are the simple roots.

RHS: principal specialisation of the Weyl-Kac character formula

LHS: comes from the construction of a basis of $L(\Lambda_0+2\Lambda_1)$ using vertex operators

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LHS: comes from the construction of a basis of $L(\Lambda_0 + 2\Lambda_1)$ using vertex operators.

Very rough idea:

- Start with a spanning set of $L(\Lambda_0 + 2\Lambda_1)$: here, monomials of the form $Z_1^{f_1} \dots Z_s^{f_s}$ for $s, f_1, \dots, f_s \in \mathbb{N}_{\geq 0}$.
- Using Lie theory, reduce this spanning set: here, it allows one to remove all monomials containing Z_j^2 or $Z_j Z_{j+1}$.
- Show that the obtained set is a basis of the representation (difficult).

Partition identities and characters

With Lepowsky and Wilson's approach (vertex operators + Weyl–Kac): discovery of many new partition identities yet unknown to combinatorialists

- Meurman–Primc 1987: higher levels of $A_1^{(1)}$
- Capparelli 1993: level 3 standard modules of $A_2^{(2)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

But often these identities are only conjectured, not proved, through this method. On the other hand, if a combinatorial proof is found, it also implies equality of characters.

Back to characters

The character

$$e^{-\lambda}\mathrm{ch}L(\lambda) = \sum_{\mu\in\mathfrak{h}^*} \dim(V_\mu)e^{\mu-\lambda}$$

is a **series with positive coefficients** in $\mathbb{Z}[[e^{-\alpha_0}, \ldots, e^{-\alpha_n}]]$
Combinatorics can help finding explicit expressions of that shape:

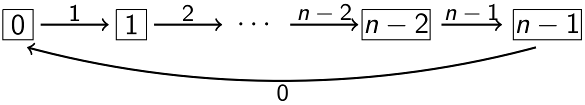
- Andrews–Schilling–Warnaar 1999
- Bartlett–Warnaar 2015
- Crystal bases (KMN^2 1992, Primc 1998, D.–Konan 2021)

Outline

Crystals: “combinatorial representations” of Lie algebras

Crystal for the vector representation of the affine Lie algebra $A_{n-1}^{(1)}$:

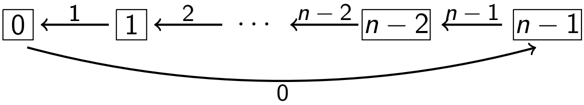
$\mathcal{B}$:



If $b_1 \xrightarrow{i} b_2$, we write $\tilde{f}_i b_1 = b_2$, or equivalently $b_1 = \tilde{e}_i b_2$.
Let $\varphi_i(b)$ (resp. $\varepsilon_i(b)$) denote the length of the maximal chain of i-arrows coming out of (resp. arriving in) b.

The dual of $\mathcal{B}$:

$\mathcal{B}^\vee$:



We have $\tilde{f}_i b_1 = b_2$ in $\mathcal{B}$ if and only if $\tilde{e}_i b_1^\vee = b_2^\vee$.

Crystals: "combinatorial representations" of Lie algebras

If $\mathcal{B}_1$ is a crystal for the representation M_1 and $\mathcal{B}_2$ is a crystal for the representation M_2, then we can define a crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ with the following arrows:

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

and $\mathcal{B}_1 \otimes \mathcal{B}_2$ is a crystal for $M_1 \otimes M_2$.

Example: $A_1^{(1)}$ at level 1

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

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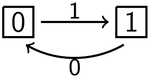
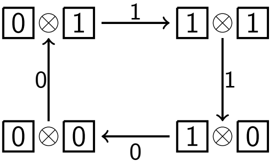


$\mathcal{B} \otimes \mathcal{B}^\vee$:



Energy functions

Definition

An *energy function* on $\mathcal{B}\otimes\mathcal{B}$ is a map $H:\mathcal{B}\otimes\mathcal{B}\to\mathbb{Z}$ satisfying for all i,

$$H\left(\tilde{e}_i(b_1\otimes b_2)\right)=\begin{cases}H(b_1\otimes b_2) & \text{if } \; i\neq 0,\\ H(b_1\otimes b_2)+1 & \text{if } \; i=0 \text{ and } \varphi_0(b_1)\geq\varepsilon_0(b_2)\\ H(b_1\otimes b_2)-1 & \text{if } \; i=0 \text{ and } \varphi_0(b_1)<\varepsilon_0(b_2).\end{cases}$$

By definition, the value of $H(b_1\otimes b_2)$ determines the values $H(b_1'\otimes b_2')$ of all the vertices $b_1'\otimes b_2'$ which are in the same connected component as $b_1\otimes b_2$.

The (KMN)^2 crystal base character formula (1992)

To each dominant weight λ, one can associate a **ground state path**

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \quad \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0,$$

where $g_i \in \mathcal{B}$ for all i.
A tensor product $\mathfrak{p} = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ of elements $p_k \in \mathcal{B}$ is said to be a λ-*path* if $p_k = g_k$ for k large enough. Let $\mathcal{P}(\lambda)$ denote the set of λ-paths .

Theorem (Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki)

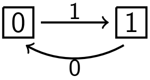
Let $L(\lambda)$ be an irreducible highest weight module of weight λ. We have

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\mathfrak{p}},$$

where $\mathrm{wt}\mathfrak{p}$ is defined in terms of the energy function and the simple roots.

Example: Primc's identity on $A_1^{(1)}$ at level 1

$\mathcal{B}$:



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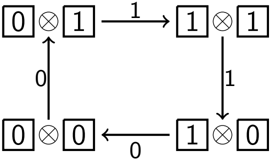

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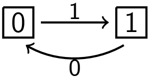

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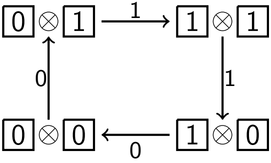

$$0 \otimes 1 \longleftrightarrow a,$$
$$0 \otimes 0 \longleftrightarrow b,$$
$$1 \otimes 1 \longleftrightarrow c,$$
$$1 \otimes 0 \longleftrightarrow d.$$

Primc's identity

Let P be the energy function in $(\mathcal{B} \otimes \mathcal{B}^\vee) \otimes (\mathcal{B} \otimes \mathcal{B}^\vee)$ for $A_1^{(1)}$.
Partitions in four colours a, b, c, d, with the order

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots,$$

and difference conditions

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

Primc (1998) conjectured that after performing the dilations

$$k_a \to 2k-1, k_b \to 2k, k_c \to 2k, k_d \to 2k+1,$$

the generating function for these partitions (not keeping track of the colours) becomes $\frac{1}{(q;q)_\infty}$.

Refinement of Primc's identity

Theorem (D.–Lovejoy 2017)

Let $P(n; k, \ell, m)$ denote the number of partitions satisfying the difference conditions of matrix P, with k parts coloured a, ℓ parts coloured c and m parts coloured d. Then

$$\sum_{n,k,\ell,m\geq 0} P(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq; q^2)_\infty(-dq; q^2)_\infty}{(q; q)_\infty(cq; q^2)_\infty}.$$

Proved via a variant of the method of weighted words (D. 2016) using q-difference equations, not at all related to crystals.

Another identity of Primc

Studying crystal bases of $A_2^{(1)}$, Primc proved that, after performing certain dilations (corresponding to the principal specialisation), the generating function for coloured partitions satisfying the difference conditions

$$
\begin{array}{c} \\ a_2b_0 \\ a_2b_1 \\ a_1b_0 \\ a_0b_0 \\ a_2b_2 \\ a_1b_1 \\ a_0b_1 \\ a_1b_2 \\ a_0b_2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} a_2b_0 & a_2b_1 & a_1b_0 & a_0b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \end{array} \\
\begin{pmatrix}
2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
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\end{pmatrix}
\end{array}
$$

becomes

$$\frac{1}{(q;q)_\infty}.$$

The starting point of our work

We wanted to generalise (purely combinatorially) Primc's two identities to obtain an infinite family of partition identities with n^2 colours.

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We wanted to generalise (purely combinatorially) Primc's two identities to obtain an infinite family of partition identities with n^2 colours.

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of symbols. We use them to define the set of colours: $\{a_i b_k : i,k \in \mathbb{N}\}$.

Definition

For all $i,k,i',k' \in \mathbb{N}$, we define the minimal difference Δ between a part coloured $a_i b_k$ and a part coloured $a_{i'} b_{k'}$ in the following way:

$$\Delta(a_i b_k, a_{i'} b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k'),$$

where $\chi(prop)$ equals 1 if *prop* is true and 0 otherwise.

For every positive integer n, let $\mathcal{P}_n$ denote the set of partitions with colours $\{a_i b_k : 0 \leq i,k \leq n-1\}$, satisfying the difference conditions Δ.

Generalisation of Primc's identity

Set for all i, $a_i = b_i^{-1}$.
Let $P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ denote the number of n^2-coloured of m which belong to $\mathcal{P}_n$, where for $i \in \{0, \ldots, n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in its colour sequence.

Theorem (D.–Konan (2019))

For every positive integer n, we have

$$\sum_{m,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1}\geq 0} P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m b_0^{v_0-u_0} \cdots b_{n-1}^{v_{n-1}-u_{n-1}}$$
$$= [x^0] \prod_{i=0}^{n-1} (-b_i^{-1}xq; q)_\infty (-b_i x^{-1}; q)_\infty .$$

Principal specialisation

In his paper, Primc used the principal specialisation:

$$\begin{cases} q & \mapsto & q^n \\ b_i & \mapsto & q^i \quad \text{for all } i \in \{0, \dots, n-1\}. \end{cases}$$

Corollary (D.–Konan (2019))

Let n be a positive integer. By doing the dilations above, the generating function for the coloured partitions in $\mathcal{P}_n$ becomes:

$$[x^0] \prod_{i=0}^{n-1} (-q^{n-i}x; q^n)_\infty (-q^i x^{-1}; q^n)_\infty = [x^0](-qx; q)_\infty (-x^{-1}; q)_\infty$$
$$= \frac{1}{(q;q)_\infty}.$$

The cases $n = 2$ and $n = 3$ recover Primc's original results.

Connection with energy functions

The difference conditions for Primc's identities were energy functions for crystals of $A_1^{(1)}$ and $A_2^{(1)}$, respectively. Are our generalised difference conditions also energy functions?

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Theorem (D.–Konan 2019)

Let n be a positive integer, and let $\mathcal{B} = \{v_i : i \in \{0, \cdots, n-1\}\}$ denote the crystal of the vector representation of $A_{n-1}^{(1)}$. The crystal $\mathbb{B} = \mathcal{B} \otimes \mathcal{B}^\vee$ is a perfect crystal of level 1 whose energy function such that $H((v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)) = 0$ satisfies for all $k, \ell, k', \ell' \in \{0, \ldots, n-1\}$,

$$H((v_{\ell'} \otimes v_{k'}^\vee) \otimes (v_\ell \otimes v_k^\vee)) = \Delta(a_k b_\ell; a_{k'} b_{\ell'}),$$

where Δ is our generalised difference condition.

Back to character formulas

Reminder: (KMN)^2 character formula

Let $L(\lambda)$ be an irreducible highest weight module of weight λ. We have

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\,\mathfrak{p}},$$

where $\mathcal{P}(\lambda)$ is the set of λ-paths.

Back to character formulas

Reminder: $(\mathrm{KMN})^2$ character formula

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where $\mathcal{P}(\lambda)$ is the set of λ-paths.

In $A_{n-1}^{(1)}$, the fundamental weights are $\Lambda_0, \ldots, \Lambda_{n-1}$. With respect to the crystal $\mathcal{B} \otimes \mathcal{B}^\vee$, they all have *constant ground state paths*.

Back to character formulas

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Goal: relate λ-paths to coloured partitions to translate our partition identities into character formulas for $A_{n-1}^{(1)}$.

Grounded partitions

Definition

Let $\mathcal{C}$ be a set of colours and $c_g \in \mathcal{C}$. Let $\succ$ be a binary relation defined on the coloured integers $\mathbb{Z}_{\mathcal{C}} = \{k_c : k \in \mathbb{Z}, c \in \mathcal{C}\}$.
A *grounded partition* with ground c_g and relation $\succ$ is a finite sequence $(\pi_0, \ldots, \pi_s)$ of coloured integers, such that

- for all $i \in \{0, \ldots, s-1\}$, $\pi_i \succ \pi_{i+1}$,
- $\pi_s = 0_{c_g}$,
- $\pi_{s-1} \neq 0_{c_g}$.

Let $\mathcal{P}_{c_g}^{\succ}$ denote the set of such partitions.

Example

Let $\mathcal{C} = c_1, c_2, c_3$, and for all $k \in \mathbb{Z}, c, c' \in \mathcal{C}$, $k_c \succ k'_{c'} \Leftrightarrow k = k' + 1$.
The sequence $(4_{c_1}, 3_{c_3}, 2_{c_2}, 1_{c_2}, 0_{c_1})$ is a grounded partition with ground c_1 and relation $\succ$.

Connection with ground state paths

Let $\mathcal{B}$ a perfect crystal and λ be a highest weight such that the corresponding ground state path is constant $\mathfrak{p}_\lambda = \cdots \otimes g \otimes g \otimes g$. Let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$ such that $H(g \otimes g) = 0$. Let $\mathcal{C}_\mathcal{B} = \{c_b : \ b \in \mathcal{B}\}$ be the set of colours indexed by the vertices of $\mathcal{B}$. We define the binary relations $\gtrdot$ and $\gg$ on $\mathbb{Z}_{\mathcal{C}_\mathcal{B}}$ by

$$k_{c_b} \gtrdot k'_{c_{b'}} \text{ if and only if } k - k' = H(b' \otimes b),$$
$$k_{c_b} \gg k'_{c_{b'}} \text{ if and only if } k - k' \geq H(b' \otimes b).$$

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Theorem (D.–Konan 2019)

The set of λ-paths is in bijection with the set of grounded partitions $\mathcal{P}^{\gtrdot}_{c_g}$.

Theorem (D.–Konan 2019)

There is a bijection between $\mathcal{P}^{\gg}_{c_g}$ and $\mathcal{P}^{\gtrdot}_{c_g} \times \mathcal{P}_{c_g}$, where $\mathcal{P}_{c_g}$ is the set of coloured partitions where all parts have colour c_g.

New combinatorial character formula

Theorem (D.–Konan 2019)

Let $L(\lambda)$ *be an irreducible highest weight module of weight* λ *with constant ground state path. Denoting by* $C(\pi)$ *the colour sequence of* π *and setting* $q = e^{-\delta/d_0}$ *and* $c_b = e^{\mathrm{wt}\, b}$ *for all* $b \in \mathcal{B}$*, we have*

$$\sum_{\pi \in \mathcal{P}_{c_g}^{\gtrdot}} C(\pi) q^{|\pi|} = e^{-\lambda} \mathrm{ch}(L(\lambda)),$$

$$\sum_{\pi \in \mathcal{P}_{c_g}^{\gg}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(L(\lambda))}{(q;q)_\infty}.$$

Non-specialised character formula for $A_{n-1}^{(1)}$

Combining our new character formula with our generalisation of Primc's identity, we obtain:

Theorem (D.–Konan 2019)

Let n be a positive integer, and let $\Lambda_0, \ldots, \Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. By setting $e^{\mathrm{wt}\, v_i} = b_i$ and $e^{-\delta} = q$, we have:

$$\frac{e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))}{(q;q)_\infty} = [x^0]\left(\prod_{i=0}^{\ell-1}(-b_i^{-1}x;q)_\infty(-b_i x^{-1}q;q)_\infty \times \prod_{i=\ell}^{n-1}(-b_i^{-1}xq;q)_\infty(-b_i x^{-1};q)_\infty\right).$$

This allows us to recover a character formula of Kac–Peterson (1984) and a new expression as a sum of infinite products with obviously positive coefficients.

Non-specialised character formula for $A_{n-1}^{(1)}$

Combining our new character formula with our generalisation of Primc's identity, we obtain:

Theorem (D.–Konan 2019)

Let $\Lambda_0, \dots, \Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. For all $\ell \in \{0, \dots, n-1\}$, we have

$$e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell)) = \left(\prod_{i=1}^{n-1} \frac{\left(e^{-i(i+1)\delta}; e^{-i(i+1)\delta}\right)_\infty}{\left(e^{-\delta}; e^{-\delta}\right)_\infty}\right) \sum_{\substack{r_1,\dots,r_{n-1}\\ r_0=r_n=0\\ 0\le r_j\le j-1}} e^{-r_l\delta} \prod_{i=1}^{n-1} e^{r_i\alpha_i} e^{r_i(r_{i+1}-r_i)\delta}$$
$$\times \left(-e^{(ir_{i+1}-(i+1)r_i-\frac{i(i+1)}{2}-\ell\chi(i\ge l>0))\delta+\sum_{j=1}^{i} j\alpha_j}; e^{-i(i+1)\delta}\right)_\infty$$
$$\times \left(-e^{((i+1)r_i-ir_{i+1}-\frac{i(i+1)}{2}+\ell\chi(i\ge l>0))\delta-\sum_{j=1}^{i} j\alpha_j}; e^{-i(i+1)\delta}\right)_\infty.$$

Outline

Multi-grounded partitions

Goal: extend the idea of grounded partitions to treat the cases of crystals where the ground state paths are not constant.

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Definition

Let $\mathcal{C}$ be a set of colors and $\succ$ a binary relation defined on $\mathbb{Z}_{\mathcal{C}}$. Suppose that there exist some colors $c_{g_0}, \ldots, c_{g_{t-1}}$ in $\mathcal{C}$ and *unique* coloured integers $u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}}$ such that

$$u^{(0)} + \cdots + u^{(t-1)} = 0,$$
$$u^{(0)}_{c_{g_0}} \succ u^{(1)}_{c_{g_1}} \succ \cdots \succ u^{(t-1)}_{c_{g_{t-1}}} \succ u^{(0)}_{c_{g_0}}.$$

Then a *multi-grounded partition* with ground $c_{g_0}, \ldots, c_{g_{t-1}}$ and relation $\succ$ is a finite sequence $\pi = (\pi_0, \cdots, \pi_{s-1}, u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}})$ of coloured integers such that $\pi_i \succ \pi_{i+1}$ for all i, and
$(\pi_{s-t}, \cdots, \pi_{s-1}) \neq (u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}})$ in terms of coloured integers.
The set of these multi-grounded partitions is denoted by $\mathcal{P}^{\succ}_{c_{g_0} \cdots c_{g_{t-1}}}$.

Example

Take $\mathcal{C} = \{c_1, c_2, c_3\}$,

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix},$$

and define the relation $\succ$ on $\mathbb{Z}_\mathcal{C}$ by $k_{c_b} \succ k'_{c_{b'}}$ if and only if $k - k' \geq M_{b,b'}$.
If we choose $(g_0, g_1) = (1, 3)$, the pair $(u^{(0)}, u^{(1)}) = (1, -1)$ is the unique pair satisfying the conditions

$$u^{(0)} + u^{(1)} = 0,$$
$$u^{(0)}_{c_1} \succ u^{(1)}_{c_3} \succ u^{(0)}_{c_1}.$$

The sequences $(3_{c_3}, 3_{c_2}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(1_{c_3}, 3_{c_1}, 1_{c_3}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ are multi-grounded partitions with ground c_1, c_3 and relation $\succ$,
$(1_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(2_{c_1}, 1_{c_1}, -1_{c_3})$ are not.

Non-constant ground state paths

Let $\mathcal{B}$ be a crystal of level ℓ, let λ be a dominant weight, and let

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0$$

be the corresponding ground state path. It is always periodic. Let t denote the period of $\mathfrak{p}_\lambda$, i.e. the smallest positive integer k such that $g_{i+k} = g_i$ for all $i \geq 0$.

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Let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$, and define

$$H_\lambda(b \otimes b') := H(b \otimes b') - \frac{1}{t} \sum_{k=0}^{t-1} H(g_{k+1} \otimes g_k).$$

Thus we have

$$\sum_{k=0}^{t-1} H_\lambda(g_{k+1} \otimes g_k) = 0.$$

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Thus we have

$$\sum_{k=0}^{t-1} H_\lambda(g_{k+1} \otimes g_k) = 0.$$

Let D be a positive integer such that $DH_\lambda(\mathcal{B} \otimes \mathcal{B}) \subset \mathbb{Z}$ and $\frac{1}{t} \sum_{k=0}^{t-1}(k+1)DH_\lambda(g_{k+1} \otimes g_k) \in \mathbb{Z}$.

Non-constant ground state paths

Let us define the relations on $\mathbb{Z}_{\mathcal{C}_\mathcal{B}}$:

$$k_{c_b} \gtrdot k'_{c_{b'}} \iff k - k' = DH_\lambda(b' \otimes b),$$
$$k_{c_b} \gg k'_{c_{b'}} \iff k - k' \geq DH_\lambda(b' \otimes b).$$

Theorem (D.–Konan 2021)

There is a bijection between the set of λ-paths $\mathcal{P}(\lambda)$ and the set ${}_t\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$ of multi-grounded partitions of $\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}}$ whose number of parts is divisible by t.

Theorem (D.–Konan 2021)

Let ${}^d\mathcal{P}$ be the set of partitions where all parts are divisible by d. There is a bijection between ${}_t\mathcal{P}^{\gtrdot}_{c_{g_0}\cdots c_{g_{t-1}}} \times {}^d\mathcal{P}$ and ${}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$, where ${}^d_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ is the set of $\pi \in {}_t\mathcal{P}^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}$ such that for all k,

$\pi_k - \pi_{k+1} - DH_\lambda(p_{k+1} \otimes p_k) \in d\mathbb{Z}_{\geq 0}$, where $c(\pi_k) = c_{p_k}$ and $\pi_s = u^{(0)}_{c_{g_0}}$.

A general character formula

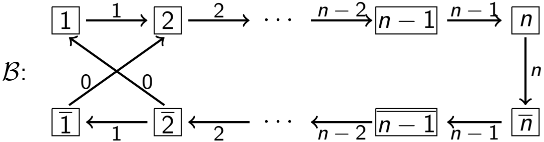
Theorem (D.–Konan 2021)

Let $L(\lambda)$ be an irreducible highest weight module of weight λ ~~with constant ground state path~~. Setting $q = e^{-\delta/(d_0 D)}$ and $c_b = e^{\overline{\mathrm{wt}}b}$ for all $b \in \mathcal{B}$, we have $c_{g_0} \cdots c_{g_{t-1}} = 1$, and the character of the irreducible highest weight module $L(\lambda)$ is given by the following expressions:

$$\sum_{\mu \in {}_t\mathcal{P}^{\gg}_{c_{g_0} \cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = e^{-\lambda} \mathrm{ch}(L(\lambda)),$$

$$\sum_{\pi \in {}^d_t\mathcal{P}^{\gg}_{c_{g_0} \cdots c_{g_{t-1}}}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(L(\lambda))}{(q^d; q^d)_\infty}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$



Ground state path: $\mathfrak{p}_{\Lambda_0} = \cdots \otimes \overline{1} \otimes 1 \otimes \overline{1} \otimes 1 \otimes \overline{1}$,

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)}(n \geq 3)$

We have $H(1 \otimes \overline{1}) + H(\overline{1} \otimes 1) = 0$, so $H_{\Lambda_0} = H$.

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We have $H(1 \otimes \overline{1}) + H(\overline{1} \otimes 1) = 0$, so $H_{\Lambda_0} = H$.

We apply our character formula with $d = 2$ and $D = 2$ and obtain

$$\sum_{\pi \in {}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}} C(\pi) q^{|\pi|} = \frac{e^{-\Lambda_0} \mathrm{ch}(L(\Lambda_0))}{(q^2; q^2)_\infty},$$

where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$.

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where $q = e^{-\delta/2}$ and $c_b = e^{\overline{\mathrm{wt}} b}$ for all $b \in \mathcal{B}$.

Thus we must compute the generating function for ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}} c_1}$, the set of multi-grounded partitions $\pi = (\pi_0, \ldots, \pi_{2s-1}, -1_{c_{\overline{1}}}, 1_{c_1})$ with relation $\gg$ and ground $c_{\overline{1}}, c_1$, **having an even number of parts**, such that for all $k \in \{0, \ldots, 2s-1\}$,

$$\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0},$$

where $c(\pi_k) = c_{p_k}$ and $\pi_{2s} = -1_{c_{\overline{1}}}$.

Example: character of Λ_0 in $A_{2n-1}^{(2)}(n \geq 3)$

$$H = \begin{array}{c|cccccccc} & 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \\ \hline 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 2 & 0 & \ddots & & & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ n & \vdots & & \ddots & \ddots & & & & \vdots \\ \overline{n} & \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ \overline{2} & 0 & 0 & & & & \ddots & \ddots & \vdots \\ \overline{1} & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{array}.$$

By the values of H, the condition $\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0}$, and the fact that $u^{(0)} = -1$, the multi-grounded partitions of ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ have parts with odd sizes.

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

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The relation $\gg$ corresponds to the following partial order on the set of coloured odd integers:

$$\begin{array}{c} (-1)_{c_{\overline{1}}} \\ 1_{c_1} \end{array} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{array}{c} 1_{c_{\overline{1}}} \\ 3_{c_1} \end{array} \ll 3_{c_2} \ll \cdots .$$

Example: character of Λ_0 in $A^{(2)}_{2n-1} (n \geq 3)$

$$H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\ \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & 1^* & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & 0^* & & \ddots & \ddots & & \vdots \\ 0 & 0 & & & & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}.$$

By the values of H, the condition $\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0}$, and the fact that $u^{(0)} = -1$, the multi-grounded partitions of ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ have parts with odd sizes.

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Only parts coloured c_1 and $c_{\overline{1}}$ can appear several times, in sequences of the form

$$\cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll \cdots \ll (2k-1)_{c_{\overline{1}}} \ll \cdots .$$

Example: character of Λ_0 in $A_{2n-1}^{(2)} (n \geq 3)$

$$\begin{matrix}(-1)_{c_{\overline{1}}} \\ 1_{c_1}\end{matrix} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{matrix}1_{c_{\overline{1}}} \\ 3_{c_1}\end{matrix} \ll 3_{c_2} \ll \cdots,$$

where parts coloured c_1 and $c_{\overline{1}}$ can repeat in sequences

$$\cdots \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll \cdots \ll (2k-1)_{c_{\overline{1}}} \ll \cdots .$$

For fixed $k \geq 1$, sequences of parts coloured c_1 and $c_{\overline{1}}$ are generated by

$$\frac{(1+c_{\overline{1}}q^{2k-1})(1+c_1q^{2k+1})}{(1-c_{\overline{1}}c_1q^{4k})}.$$

For $k=0$, the sequence $(1_{c_1},(-1)_{c_{\overline{1}}},1_{c_1})$ can occur at the end of the partitions grounded in $c_{\overline{1}}, c_1$, but $((-1)_{c_{\overline{1}}},1_{c_1},(-1)_{c_{\overline{1}}},1_{c_1})$ cannot.
So, if we temporarily forgot the condition on the even number of parts in ${}^2_2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$, the generation function would be

$$(1+c_1q)\cdot\frac{(-c_1q^3,-c_{\overline{1}}q,-c_2q,-c_{\overline{2}}q,\ldots,-c_nq,-c_{\overline{n}}q;q^2)_\infty}{(c_{\overline{1}}c_1q^4;q^4)_\infty}.$$

Example: character of Λ_0 in $A_{2n-1}^{(2)}(n \geq 3)$

Observation

$$\sum_{n,k\geq 0} a_{n,k}x^k q^n + \sum_{n,k\geq 0} a_{n,k}(-x)^k q^n = 2\sum_{n,k\geq 0} a_{n,2k}x^{2k}q^n$$

Thus, the generating function for multi-grounded partitions in ${}_2^2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}$ is

$$\sum_{\pi\in {}_2^2\mathcal{P}^{\gg}_{c_{\overline{1}}c_1}} C(\pi)q^{|\pi|} = \frac{1}{2(c_{\overline{1}}c_1q^4;q^4)_\infty}\Big((-c_1q,-c_{\overline{1}}q,\dots,-c_nq,-c_{\overline{n}}q;q^2)_\infty + (c_1q,c_{\overline{1}}q,\dots,c_nq,c_{\overline{n}}q;q^2)_\infty\Big)$$

$$= \frac{e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0))}{(q^2;q^2)_\infty},$$

where $\delta = \alpha_0+\alpha_1+2\alpha_2\cdots+2\alpha_{n-1}+\alpha_n$,
$q = e^{-\delta/2}$ and $c_i = e^{\alpha_i+\cdots+\alpha_{n-1}+\alpha_n/2}$ for all $i \in \{1,\dots,n\}$.

Thank you very much!