Symmetric functions and interacting particle processes

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joint with Arvind Ayyer and James Martin,
arXiv:2011.06117, arXiv:2209.09859

particle models and symmetric functions

1. the asymmetric simple exclusion process (ASEP) $\rightarrow$ combinatorial formula for Macdonald polynomials and some nice specializations
2. modified Macdonald polynomials $\rightarrow$ the multispecies totally asymmetric zero range process (mTAZRP) and observables

exactly solvable interacting particle models

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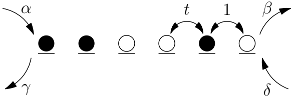
- we are interested in studying integrable systems whose exact solutions (e.g. stationary distributions) can be expressed in terms of combinatorial formulas or special functions (e.g. Macdonald polynomials)
- the field was initiated by Spitzer in his 1970 paper where he defined the ASEP (Asymmetric Simple Exclusion Process) and the ZRP (Zero Range Process)

Asymmetric Simple Exclusion Process (ASEP)



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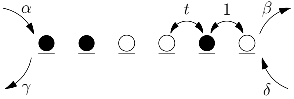

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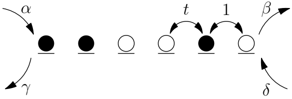

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single species ASEP

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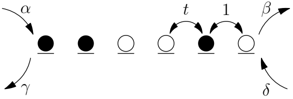

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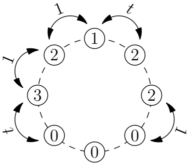
- **dynamics:** any two adjacent particles may swap with some predetermined rate (in our case, fixed by a parameter $0 \leq t \leq 1$):

$$XABY \xrightarrow{1} XBAY \quad \text{and} \quad XBAY \xrightarrow{t} XABY \quad \text{for} \quad A > B$$

our setting: ASEP on a circle



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- Fix a circular lattice on n sites, and choose n nonnegative integer weights. Collection of weights: a vector $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.

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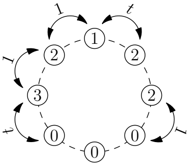

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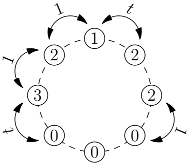

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- The transitions are swaps of adjacent particles $A > B$ (fix $0 \leq t \leq 1$):

$$X \; \textcircled{A} \; \textcircled{B} \; Y \underset{t}{\overset{1}{\rightleftharpoons}} X \; \textcircled{B} \; \textcircled{A} \; Y$$

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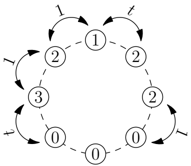

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- For example, ASEP($(2, 2, 1, 0)$) has 12 states:

$$(2, 2, 1, 0), (2, 1, 2, 0), (2, 1, 0, 2), (2, 2, 0, 1), (2, 0, 2, 1), (2, 0, 1, 2), (0, 2, 2, 1), \cdots$$

The transitions from state $(2, 1, 2, 0)$ are:

- $(1, 2, 2, 0)$ with probability $t/4$
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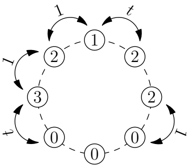

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- Goal: compute the stationary probabilities

Example for $\lambda = (2,1)$ and $n = 3$

- the elements in the state space are:
  ② ① ⓪ ② ⓪ ① ① ② ⓪ ① ⓪ ② ⓪ ② ① ⓪ ① ②

- the (row stochastic) transition matrix is:

$$\begin{pmatrix} 1-\frac{2+t}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{t}{3} \\ \frac{t}{3} & 1-\frac{1+2t}{3} & 0 & \frac{t}{3} & \frac{1}{3} & 0 \\ \frac{t}{3} & 0 & 1-\frac{1+2t}{3} & \frac{1}{3} & \frac{t}{3} & 0 \\ 0 & \frac{1}{3} & \frac{t}{3} & 1-\frac{2+t}{3} & 0 & \frac{1}{3} \\ 0 & \frac{t}{3} & \frac{1}{3} & 0 & 1-\frac{2+t}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{t}{3} & 0 & \frac{t}{3} & 1-\frac{1+2t}{3} \end{pmatrix}$$

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- the (unnormalized) stationary distribution is:

$$\widetilde{\Pr}((2,1,0)) = \widetilde{\Pr}((1,0,2)) = \widetilde{\Pr}((0,2,1)) = 2+t$$
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From ASEP to Macdonald polynomials

Define the partition function of ASEP(λ, n):

$$\mathcal{Z}_{\lambda,n}(t) = \sum_{\alpha \in S_n \cdot \lambda} \widetilde{\Pr}(\alpha)(t).$$

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Theorem (Cantini–de Gier–Wheeler '15)

The partition function of ASEP(λ, n) *is a specialization of the Macdonald polynomial:*

$$P_\lambda(1, \ldots, 1; 1, t) = \mathcal{Z}_{\lambda,n}(t)$$

Symmetric functions

- Let $X = x_1, x_2, \cdots$ be a family of indeterminates, and let $\Lambda = \Lambda_{\mathbb{Q}}$ be the algebra of symmetric functions in X over $\mathbb{Q}$
  - $f(x_1, \ldots, x_n) \in \Lambda$ is symmetric if $\forall \pi \in S_n$, $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$

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- Λ has several nice bases: e.g. $\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, indexed by partitions λ.
  E.g. $m_{(2,1)} = \sum_{i,j} x_i^2 x_j^1 = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots$

Let $\langle , \rangle$ be the standard inner product on Λ. $\{s_\lambda\}$ is the unique basis of Λ:

i. orthogonal with respect to $\langle , \rangle$

ii. upper triangular with respect to $\{m_\lambda\}$:

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- $s_\lambda = \sum_\sigma x^\sigma$ where σ's are semi-standard fillings of the Young diagram of shape λ

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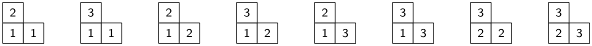

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- Example:

$$P_{(2,1)}(X; q, t) = m_{(2,1)} + \frac{(1 - t)(2 + q + t + 2qt)}{1 - qt^2} m_{(1,1,1)}.$$

Combinatorial formulas

- Haglund-Haiman-Loehr '04 gave a formula for P_λ as a sum over tableaux with statistics maj and (co)inv:

$$P_\lambda(X;q,t) = \sum_{\substack{\sigma \in \mathrm{dg}(\lambda) \\ \sigma \text{ non-attacking}}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_u \frac{1-t}{1-q^{\mathrm{leg}(u)+1} t^{\mathrm{arm}(u)+1}}$$

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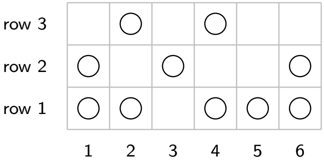
- Corteel-M-Williams '18: a new formula for P_λ in terms of multiline queues, which also give formulas for the stationary distribution of the ASEP; this was inspired by the result of Cantini-de Gier-Wheeler '15

multiline queues

- a multiline queue (MLQ) of type λ, n is an arrangement and pairing of balls on a $n \times \lambda_1$ lattice, with λ'_j balls in row j.



$$n = 6$$

$$\lambda = (3, 3, 2, 1, 1)$$

$$\lambda' = (2, 3, 5)$$

Angel '08, Ferrari-Martin '07 ($t = 0$ case), Martin '18 (for $q = x_1 = \cdots = x_n = 1$), Corteel–M–Williams '18 (general)

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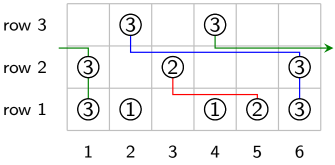

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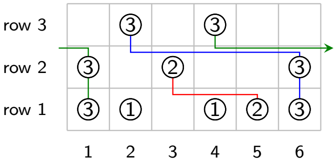

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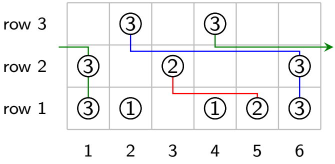

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- The weight $\mathrm{wt}(M)$ of a multiline queue depends on the parameters $t, q, x_1, \ldots, x_n$:

$$\text{weight} = x_1^2 x_2^2 x_3 x_4^2 x_5 x_6^2 q t^2 \frac{(1-t)^3}{(1-qt^3)^2(1-qt^2)}$$

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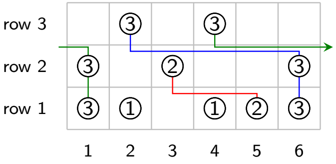

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2	4			
6	1	3		
6	1	5	2	3

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- Can be represented by a tableau, where each string is mapped to a column

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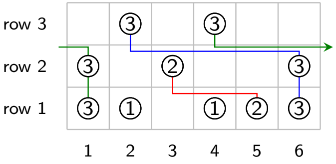

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2	4			
6	1	3		
6	1	5	2	3

Angel '08, Ferrari-Martin '07 ($t = 0$ case), Martin '18 (for $q = x_1 = \cdots = x_n = 1$), Corteel–M–Williams '18 (general)

- The state of a multiline queue is read off the bottom row
- The weight $\mathrm{wt}(M)$ of a multiline queue depends on the parameters $t, q, x_1, \ldots, x_n$:

$$\text{weight} = x_1^2 x_2^2 x_3 x_4^2 x_5 x_6^2 q t^2 \frac{(1-t)^3}{(1-qt^3)^2(1-qt^2)}$$

- Can be represented by a tableau, where each string is mapped to a column
- Can be represented by a queueing system and described as a coupled system of 1-ASEPs. The pairing is a projection map onto the n-ASEP.

From ASEP to Macdonald polynomials

Theorem (Martin '18, Corteel-M-Williams '18)

The (unnormalized) stationary probability of state α *of the mASEP is*

$$\widetilde{\Pr}(\alpha)(t) = \sum_{M \in \mathrm{MLQ}(\alpha)} wt(M)(1, \ldots, 1; 1, t)$$

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Theorem (Cantini–de Gier–Wheeler '15)

The partition function of $\mathrm{ASEP}(\lambda, n)$ *is a specialization of the Macdonald polynomial:*

$$P_\lambda(1, \ldots, 1; 1, t) = \mathcal{Z}_{\lambda,n}(t) = \sum_{\alpha \in S_n \cdot \lambda} \widetilde{\Pr}(\alpha)(t).$$

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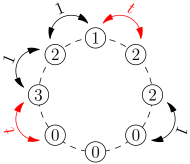
Theorem (Corteel–M–Williams '18)

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{M \in \mathrm{MLQ}(\lambda, n)} \mathrm{wt}(M)(x_1, \ldots, x_n; q, t)$$

(also Lenart '09 for λ with distinct parts.)

$t = 0$: Hall-Littlewood polynomials $P_\lambda(X; q, 0)$ via multiline queues

- at $t = 0$, the ASEP becomes the TASEP:



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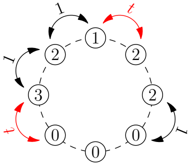

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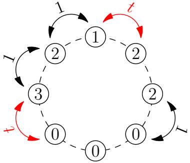
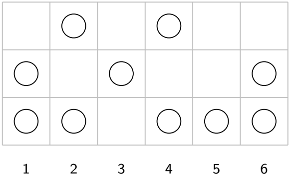


- Define the Hall-Littlewood polynomial $P_\lambda(X; q) = P_\lambda(X; q, 0)$
- In the multiline queues, pairings become forced.



$n = 6$

$\lambda = (3, 3, 2, 1, 1)$

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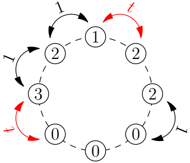

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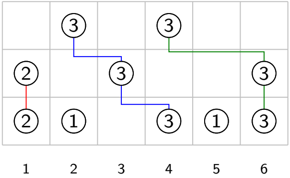

$n = 6$

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$\alpha = (2, 1, 0, 3, 1, 3)$

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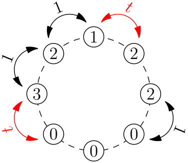


- Define the Hall-Littlewood polynomial $P_\lambda(X; q) = P_\lambda(X; q, 0)$
- In the multiline queues, pairings become forced.

	3		3		
2		3			3
2	1		3	1	3
1	2	3	4	5	6

$n = 6$

$\lambda = (3, 3, 2, 1, 1)$

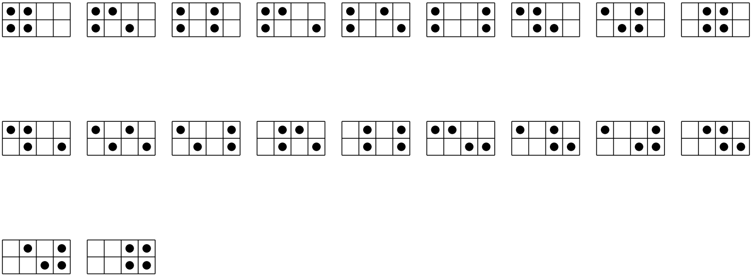
$\alpha = (2, 1, 0, 3, 1, 3)$

- call $\mathrm{MLQ}(\lambda, n)$ the set of ball arrangements with λ'_j balls in each row j, on a lattice of size $n \times \lambda_1$. (The labels can be recovered uniquely)

$$M = (\{1, 2, 4, 5, 6\}, \{1, 3, 6\}, \{2, 4\})$$

$q = t = 0$: Schur polynomials via multiline queues

- at $q = 0$, the multiline queues are non-wrapping, denote this set by $\mathrm{MLQ}_0(\lambda, n)$:



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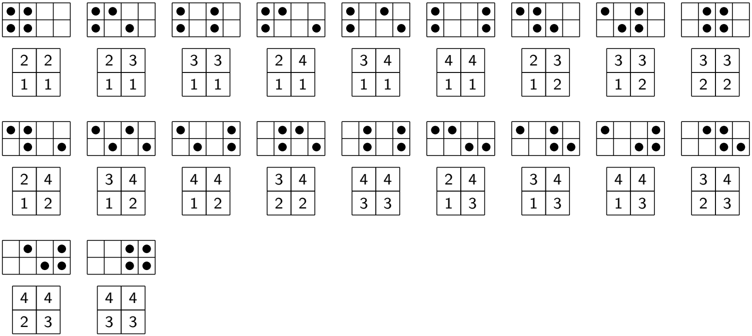

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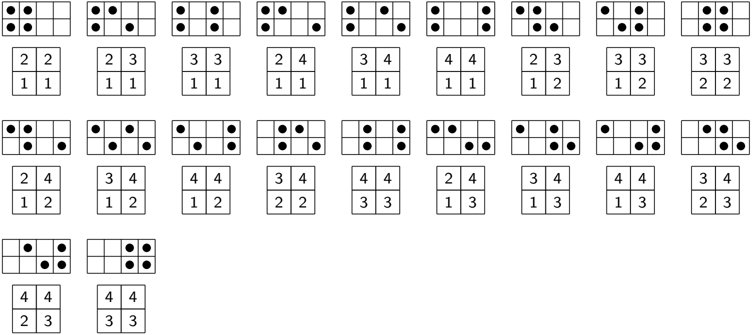


- the map $\mathrm{MLQ}_0(\lambda) \to \mathrm{SSYT}(\lambda)$ is given by column RSK applied to the row reading word of the multiline queue. (bottom to top, left to right)

Lascoux–Schützenberger charge formula via MLQs

• For a permutation σ, define $\text{charge}(\sigma) := \text{maj}(\text{rev}(\sigma^{-1}))$. For a SSYT τ, $\text{charge}(\tau) = \text{charge}(\text{rw}(\tau))$.

Theorem (Lascoux–Schützenberger,'78)

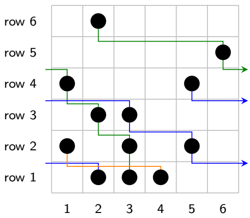
$$P_\lambda(X;q,0) = \sum_{\mu \le \lambda} K_{\mu'\lambda'}(q) s_\mu, \qquad K_{\lambda\mu}(q) = \sum_{Q \in \text{SSYT}(\lambda,\mu)} q^{\text{charge}(Q)}$$

- Define collapsing procedure ρ (with Jerónimo Valencia '23+):

$$\begin{aligned} \text{MLQ}(\lambda) &\longrightarrow \bigcup_\mu \text{MLQ}_0(\mu) \times \text{SSYT}(\mu', \lambda') \\ M &\longrightarrow (M_0, Q) \end{aligned}$$

- $\text{charge}(Q) = \text{maj}(M)$ (the q-statistic, keeps track of wrapping pairings)
- can be described using lowering operators on the column reading word of M
- lifting procedure ρ^{-1} can be described using raising operators
- generalizes to a quasisymmetric refinement of $K_{\lambda\mu}(q)$.

Collapsing procedure



$Q = \emptyset$

$M \in \mathrm{MLQ}((6,4,2))$

$\mathrm{maj}(M) = 2 + 1 + 3 = 6$

Collapsing procedure

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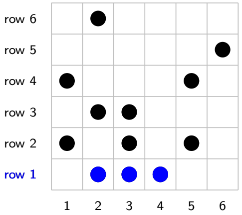

$$Q = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$$

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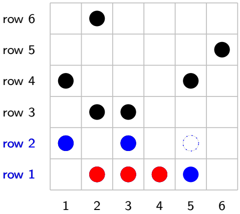
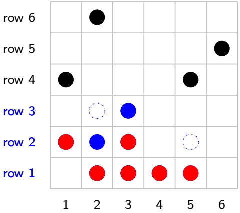


$Q =$

2	2		
1	1	1	2

Collapsing procedure



$Q =$

3			
2	2	3	
1	1	1	2

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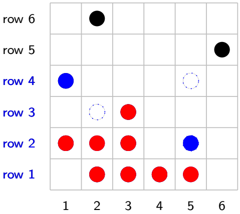

$Q =$

4			
3			
2	2	3	4
1	1	1	2

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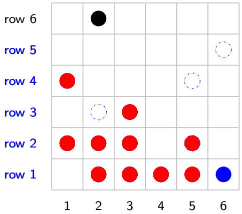

$Q =$

4				
3				
2	2	3	4	
1	1	1	2	5

Collapsing procedure

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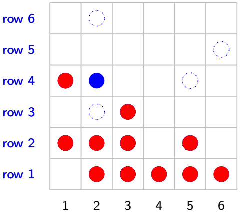
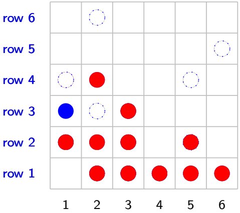


$Q =$

4				
3				
2	2	3	4	
1	1	1	2	5

Collapsing procedure



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3	6			
2	2	3	4	
1	1	1	2	5

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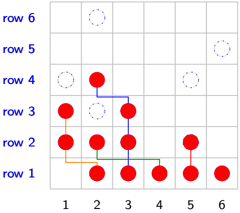
Collapsing procedure



$$Q = \begin{array}{|c|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 3 & 6 \\ \cline{1-4} 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 2 & 5 \\ \hline \end{array}$$

$$M \in \mathsf{MLQ}((6,4,2))$$

$$\mathsf{maj}(M) = 2 + 1 + 3 = 6$$

$$M_0 \in \mathsf{MLQ}_0((4,3,2,2,1))$$

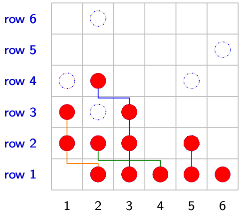
$$\mathsf{charge}(Q) = \mathsf{charge}(4|36|2234|11125)$$

$$= \mathsf{charge}(436215) + \mathsf{charge}(3412) + \mathsf{charge}(21)$$

$$= 2 + (1+3) + 0$$

$$Q \in \mathsf{SSYT}(\mu', \lambda')$$

Collapsing procedure



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4				
3	6			
2	2	3	4	
1	1	1	2	5

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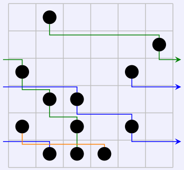
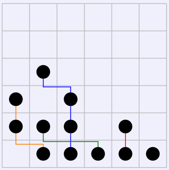


$\xrightarrow{\rho}$

$\Big($



,

4				
3	6			
2	2	3	4	
1	1	1	2	5

$\Big)$

$\lambda = (6,4,2)$
$\mathrm{maj}(M) = 6$

$\lambda = (4,3,2,2,1)$
$\mathrm{maj}(M_0) = 0$

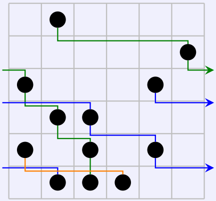
$\mathrm{charge}(Q) = 6$

Collapsing procedure

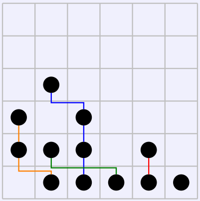
Lemma

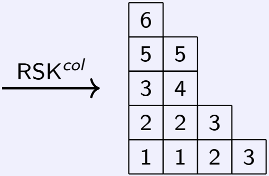
$$\mathrm{RSK}^{col} : \binom{a_1, \ldots, a_n}{b_1, \ldots, b_n} \to \mathrm{SSYT}(\mu') \times \mathrm{SSYT}(\mu)$$

$$\mathrm{RSK}^{col}(\mathrm{rw}(M)) = \mathrm{RSK}^{col}(\mathrm{rw}(\rho(M)))$$



and





$\mathrm{rw}(M) = 234|135|23|15|6|2$ $\quad$ $\mathrm{rw}(\rho(M)) = 23456|1235|13|2$

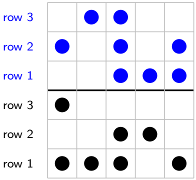
Littlewood-Richardson rule via MLQs

Theorem

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu = \sum_\nu c^{\nu'}_{\lambda'\mu'} s_\nu$$

- Using the collapsing procedure:

$$\mathsf{MLQ}_0(\lambda) \times \mathsf{MLQ}_0(\mu) \to \mathsf{MLQ}_0(\nu) \times \mathsf{SSYT}^*(\nu'/\lambda', \mu)$$



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$$M_1 \in \mathsf{MLQ}(\lambda), \qquad \lambda = (4,2,1)$$
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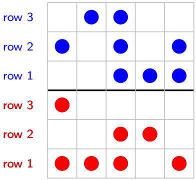

$$Q = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 1 & 1 & 1 & 1 \\ \hline \end{array}$$

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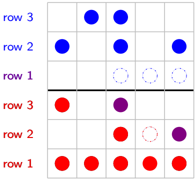

$$Q = \begin{array}{|c|c|c|c|c|} \hline 3 & 1 \\ \cline{1-3} 2 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

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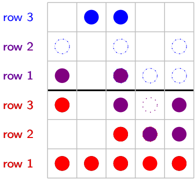

$$Q = \begin{array}{|c|c|c|c|c|}\hline 2 & 2 \\ \cline{1-2} 3 & 1 & 2 \\ \cline{1-3} 2 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}$$

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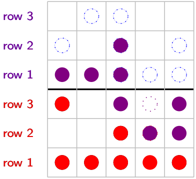

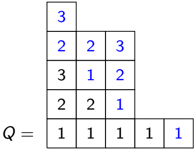

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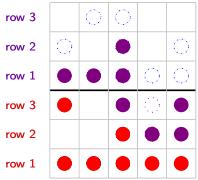

$Q =$

3				
2	2	3		
3	1	2		
2	2	1		
1	1	1	1	1

$$\begin{aligned} M_1 &\in \mathsf{MLQ}(\lambda), & \lambda &= (4,2,1) \\ M_2 &\in \mathsf{MLQ}(\mu), & \mu &= (3,3,2) \\ M &\in \mathsf{MLQ}(\nu), & \nu &= (5,4,4,1,1) \end{aligned}$$

$$Q \in \mathsf{SSYT}^*(\nu'/\lambda', \mu')$$

- $\mathsf{SSYT}^*(\nu'/\lambda', \mu')$ is the set of Yamanouchi fillings of ν'/λ' with content μ'

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Example: $\widetilde{H}_{(2,1)}(X; q, t) = m_{(3)} + (1 + q + t)m_{(2,1)} + (1 + 2q + 2t + qt)m_{(1,1,1)}$

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From multiline queues to a new formula for $\widetilde{H}_\lambda$

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- the resulting objects are of the same flavor as multiline queues, except that multiple balls are allowed at each location. (This translates to removing the "non-attacking" condition from the corresponding tableaux)

Example: $\widetilde{H}_{(2,1)}(X;q,t)$

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m_3	$q\, m_{21}$	$t\, m_{21}$	m_{21}	$t\, m_{111}$	$q\, m_{111}$	$qt\, m_{111}$	$t\, m_{111}$	m_{111}	$q\, m_{111}$

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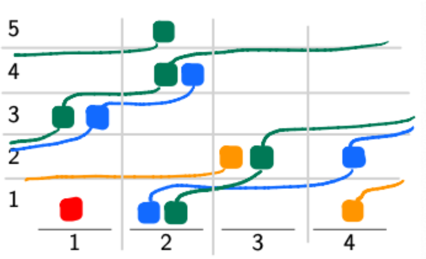
motivation for queue inversions: multiline diagrams

The tableaux are actually representing a queueing system which is an arrangement of lattice paths/strings

2			
2	2		
1	1		
3	4	3	
2	2	4	1

⟷



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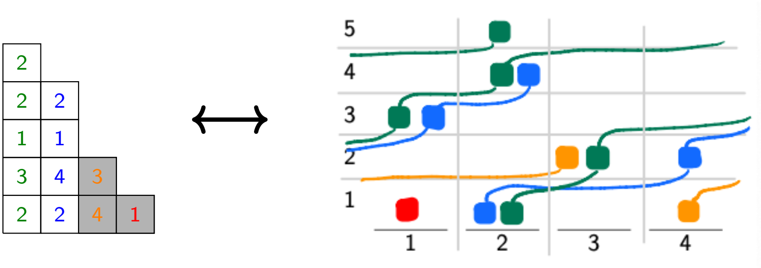

“plethystic version” of certain non-attacking fillings ⟷ “plethystic version” of multiline queues

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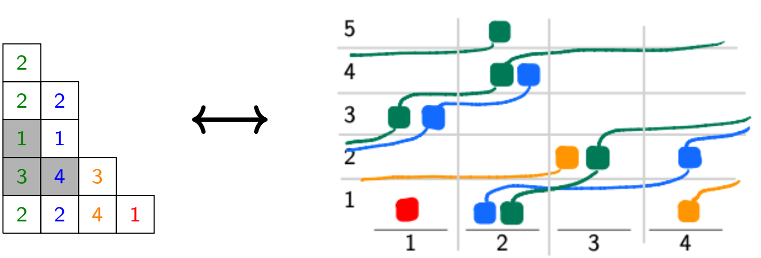
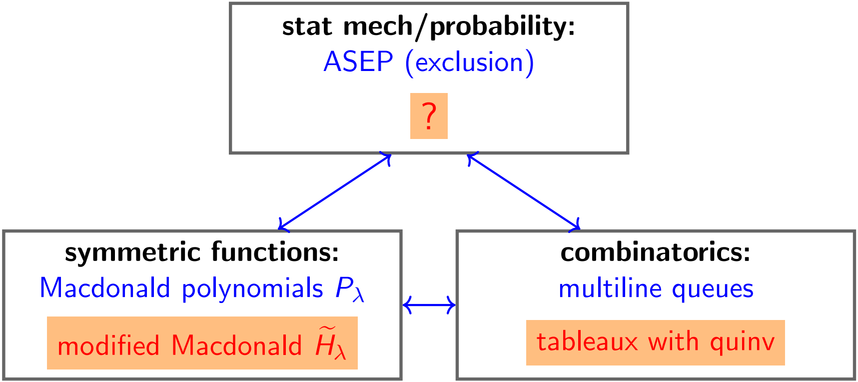


"plethystic version" of certain non-attacking fillings ⟷ "plethystic version" of multiline queues

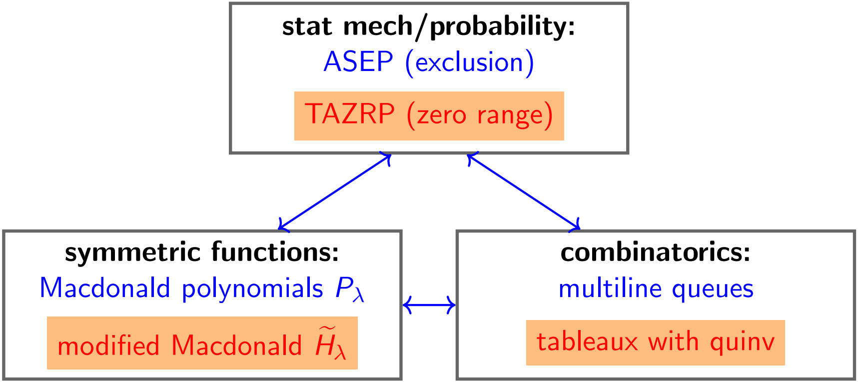
queue inversion ⟷ skipped particle

Big picture



What is the analogous interacting particle system whose partition function is a specialization of $\widetilde{H}_\lambda$?

Big picture



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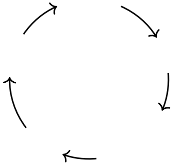
totally asymmetric zero range processes (TAZRP)

- continuous-time stochastic processes (Spitzer '70), can be defined on arbitrary graphs. In our case, we have a circular lattice with n sites.



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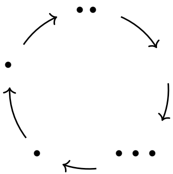

Here, $n = 5$, $k = 7$

$$\tau = (\ 11 \mid \cdot \mid 111 \mid 1 \mid 1\ )$$

- simplest case: there are k indistinguishable particles, moving **clockwise**. A configuration $\tau = (\tau_1, \ldots, \tau_n)$ is any allocation of the k particles on the n sites.

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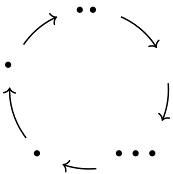

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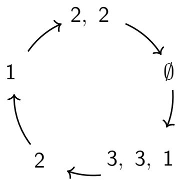

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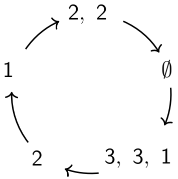

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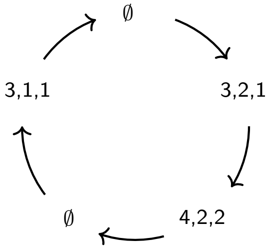
- Kuniba–Maruyama–Okado (2015+) (and others) have studied many multispecies variants of the TAZRP. **All of these are integrable!** The version we will describe was first studied by Takayama '15

the mTAZRP: states

- Fix a (circular 1D) lattice on n sites and a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ for the particle types



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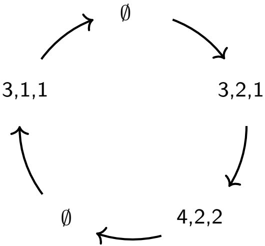
- TAZRP(λ, n) is a Markov chain whose states are *multiset compositions* τ of type λ, with n (possibly empty) parts



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$$f_j(k) = x_j^{-1} t^d \sum_{u=0}^{c-1} t^u$$

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For example: If site j contains the particles $\{4, 3, 3, 1, 1, 1\}$, then:

$$\begin{aligned}
k &= 1: & d &= 3, & c &= 3, & f_j(1) &= x_j^{-1} t^3 (1 + t + t^2). \\
k &= 3: & d &= 1, & c &= 2, & f_j(3) &= x_j^{-1} t (1 + t). \\
k &= 4: & d &= 0, & c &= 1, & f_j(4) &= x_j^{-1}.
\end{aligned}$$

Lumping of tableaux to mTAZRP

Very similar projection map as for the ASEP.

- Given a filling σ, read the state $\tau \in \mathsf{TAZRP}(\lambda, n)$ from the bottom row of σ as follows:

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$$\tau_j \text{ is the multiset } \{\lambda_i : \sigma(1, i) = j\}$$

- For example, for $\lambda = (2, 1, 1)$ and $n = 3$, the following are all the tableaux that correspond to the state $\tau = \left(21 \mid \cdot \mid 1 \right)$:

$$\begin{array}{|c|cc} \hline 1 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

TAZRP probabilities and tableaux

Theorem (Ayyer–M–Martin '21)

Fix λ, n. The (unnormalized) stationary probability of $\tau \in \mathrm{TAZRP}(\lambda, n)$ is

$$\widetilde{\Pr}(\tau) = \sum_{\substack{\sigma : \mathrm{dg}(\lambda) \to [n] \\ \sigma \text{ has type } \tau}} x^{\sigma} t^{\mathrm{quinv}(\sigma)}.$$

Corollary

The so-called partition function of $\mathrm{TAZRP}(\lambda, n)$ *is*

$$\mathcal{Z}_{\lambda,n}(x_1, \ldots, x_n; t) = \widetilde{H}_{\lambda}(x_1, \ldots, x_n; 1, t).$$

Proof: construction of a Markov chain on tableaux that lumps to the TAZRP.

an example for $\lambda = (2,1,1)$ and $n = 2$

The stationary distribution is:

$(\,211 \mid \cdot\,)$	$x_1^3(x_1+x_2)$
$(\,11 \mid 2\,)$	$x_1^2x_2(t^2x_2+x_1)$
$(\,21 \mid 1\,)$	$x_1^2x_2(tx_1+x_2)(1+t)$
$(\,1 \mid 21\,)$	$x_1x_2^2(x_1+tx_2)(1+t)$
$(\,2 \mid 11\,)$	$x_1x_2^2(t^2x_1+x_2)$
$(\,\cdot \mid 211\,)$	$x_2^3(x_1+x_2)$

Example computation for $(\,21 \mid 1\,)$:

$$\begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & 1 & 2 \\ \hline\end{array} : t^2, \qquad \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & 1 & 2 \\ \hline\end{array} : t, \qquad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & 2 & 1 \\ \hline\end{array} : t, \qquad \begin{array}{|c|cc}\hline 2 & & \\ \hline 1 & 2 & 1 \\ \hline\end{array} : 1$$

the total is: $\widetilde{\Pr}(21|1) = x_1^2x_2(tx_1+x_2)(1+t)$.

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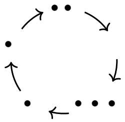

Here, $n = 5$, $m = 7$

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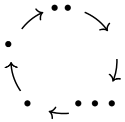

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- The stationary probability of the configuration τ is:

$$\pi(\tau) = \frac{1}{\widetilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)} \begin{bmatrix} m \\ \tau_1, \ldots, \tau_n \end{bmatrix}_t \prod_{i=1}^{n} x_i^{\tau_i}$$

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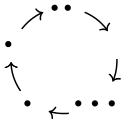

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Proposition (Current for the single species TAZRP)

For the single-species TAZRP on n sites with m particles, the current is given by

$$J = [m]_t \, \frac{\widetilde{H}_{(1^{m-1})}(x_1, \ldots, x_n; 1, t)}{\widetilde{H}_{(1^m)}(x_1, \ldots, x_n; 1, t)}.$$

Current

Theorem (Ayyer-M-Martin '22+)

Let $\lambda = (1^{m_1}, \ldots, k^{m_k})$*, and let* $1 \leq j \leq k$*. The current of the particle of type* j *of the TAZRP of type* λ *on* n *sites is given by*

$$J = \left[m_j + \cdots + m_k\right]_t \frac{\widetilde{H}_{\left(1^{m_j + \cdots + m_k - 1}\right)}}{\widetilde{H}_{\left(1^{m_j + \cdots + m_k}\right)}} - \left[m_{j+1} + \cdots + m_k\right]_t \frac{\widetilde{H}_{\left(1^{m_{j+1} + \cdots + m_k - 1}\right)}}{\widetilde{H}_{\left(1^{m_{j+1} + \cdots + m_k}\right)}}$$

Densities

- Take TAZRP(λ, n) with content $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$.
- Define $z_j^{(\ell)}$ to be the random variable counting the number of particles of type ℓ at site j in a configuration of TAZRP(λ, n).
- Denote the expectation in the stationary distribution by $\langle z_j^{(\ell)} \rangle$.

Theorem (Ayyer-M-Martin '22+)

For $1 \leq \ell \leq k$, the density of the ℓ'th species at site 1 is given by

$$\langle z_1^{(\ell)} \rangle = x_1 \partial_{x_1} \log \left(\frac{\widetilde{H}_{(1^{m_\ell + \cdots + m_k})}(x_1, \ldots, x_n; 1, t)}{\widetilde{H}_{(1^{m_{\ell+1} + \cdots + m_k})}(x_1, \ldots, x_n; 1, t)} \right).$$

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Corollary

$\langle z_1^{(\ell)} \rangle$ *is symmetric in the variables* $\{x_2, \dots, x_n\}$.

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Theorem (Ayyer-M-Martin '22)

$\mathbb{P}_{\lambda,n}(\overline{w})$ is tcbsymmetric in the variables $\{x_{\ell+1}, \dots, x_n\}$.

final remarks

- ~~Explicit bijection from the inv to the quinv statistic?~~

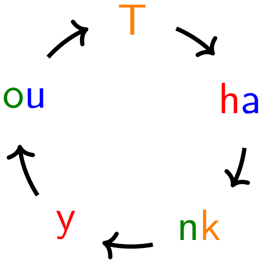
(recently found by Loehr)

- Can we find a dynamical process that incorporates the q as a parameter?

  This seems difficult because

  - We lose factorization of $\widetilde{H}_\lambda$
  - We lose translation invariance

- Suitable quasisymmetric version of modified Macdonald polynomials? Nonsymmetric version?



Modified Macdonald polynomials and the multispecies zero range process:
arXiv:2011.06117, arXiv:2209.09859