UNDERSTANDING DYSON-SCHWINGER EQUATIONS VIA CHORD DIAGRAMS

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Univ. de Caen Normandie

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Joint work with Karen YEATS (Univ. of Waterloo)

In this work, analytic combinatorics applied to

Q.F.T

In this work, analytic combinatorics applied to

QUANTUM FIELD THEORY

Part I

Background about Q.F.T.

In this work, analytic combinatorics applied to

QUANTUM FIELD THEORY

In this work, analytic combinatorics applied to

QUANTUM FELINE THEORY

~~FIELD~~

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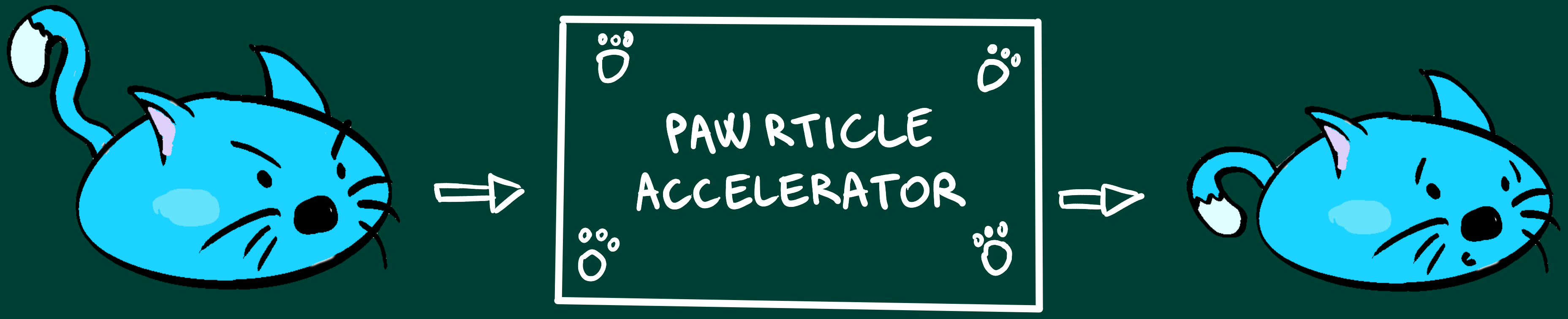
PHYSICAL BACKGROUND

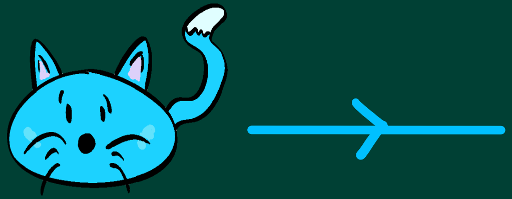
Yukawa theory



PHYSICAL BACKGROUND

Yukawa theory

Feynman diagram



fermion

meson

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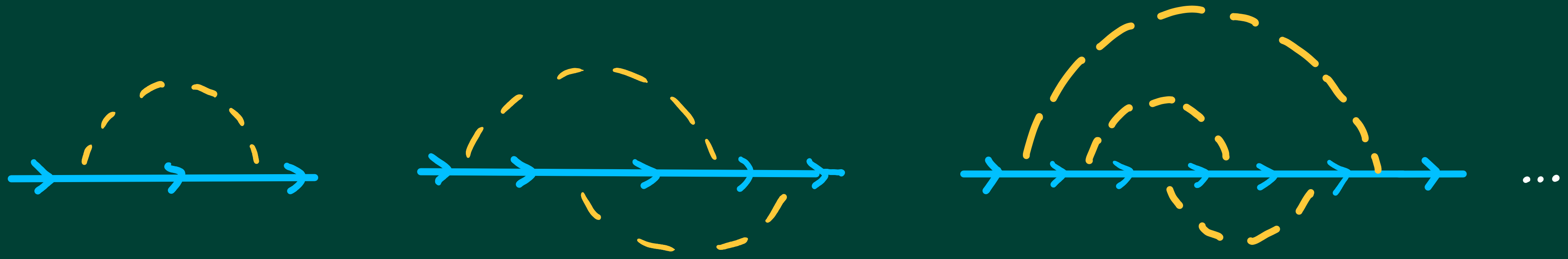
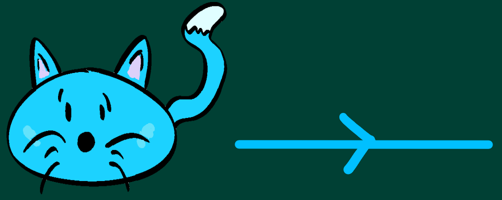


Feynman diagrams



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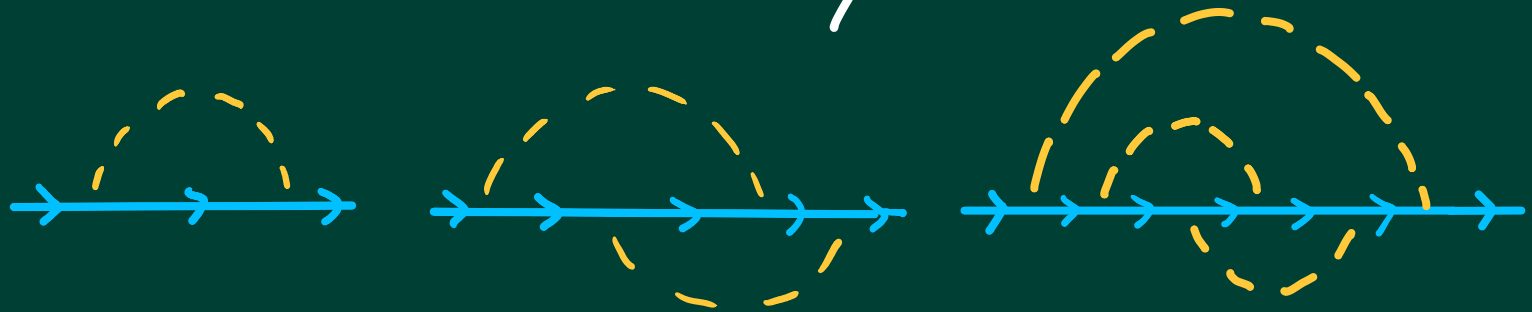

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PHYSICAL BACKGROUND

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Feynman diagrams

renormalized Feynman integrals

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Yukawa theory

Feynman diagrams

renormalized Feynman integrals

perturbative series (Green function)

$$G(x, L) = \sum_{D \text{ Feynman diagram}} M_D(L)\, x^{\#\text{vertices}(D)}$$

L = log of momentum

x = coupling constant

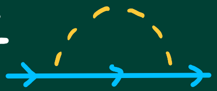
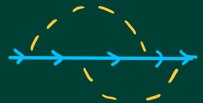
G(x, L) = probability amplitude

DYSON - SCHWINGER EQUATION

The equation (which is a Dyson - Schwinger equation)

$$G(x,L) = 1 - \sum_{k \geq 1} x^{k}\, G(x,\partial_{-p})^{1-2k} \left(e^{-Lp}-1\right) F_k(p) \Big|_{p=0}$$

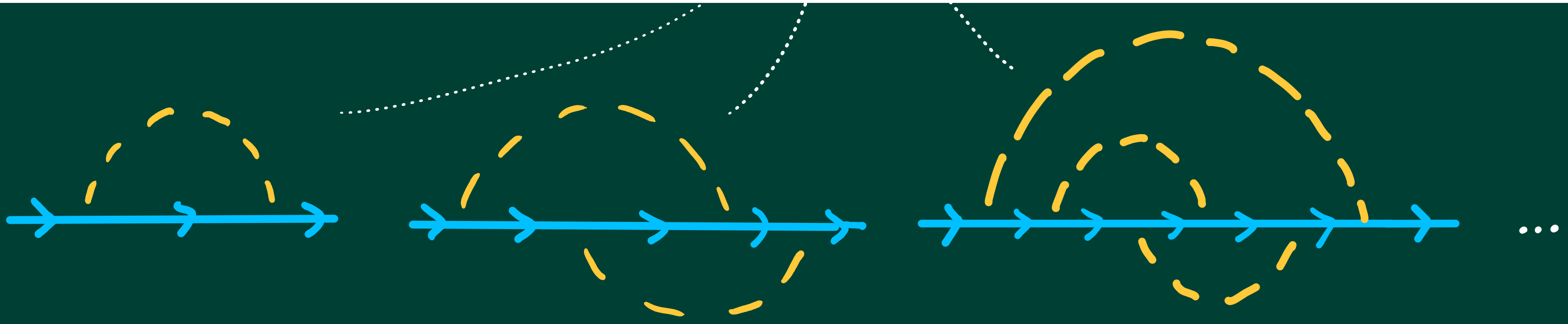
where $F_k(p)$ is the regularized Feynman integral of the primitive graphs of size k

($F_1(p)$ = contribution of  , $F_2(p)$ = contribution of  +  , ...)

has for solution

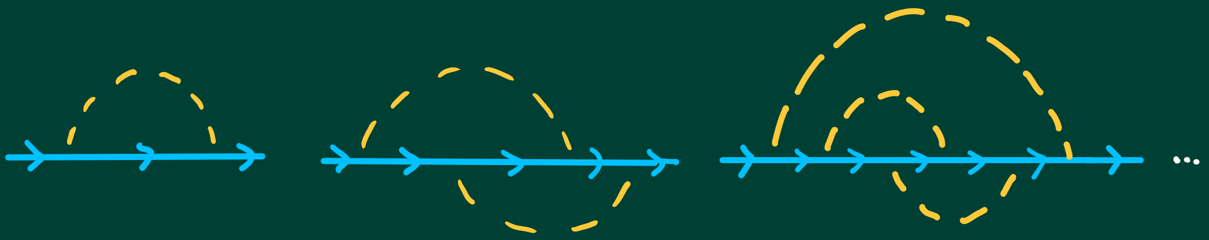
$$G(x,L) = \sum_{\substack{D \text{ Feynman} \\ \text{diagram}}} \mathcal{M}_D(L)\, x^{\#\text{vertices}(D)}$$

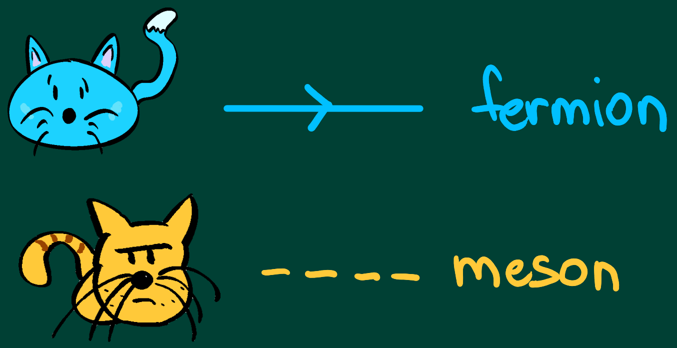
Feynman diagrams



PHYSICAL BACKGROUND

Yukawa theory : not the only theory!





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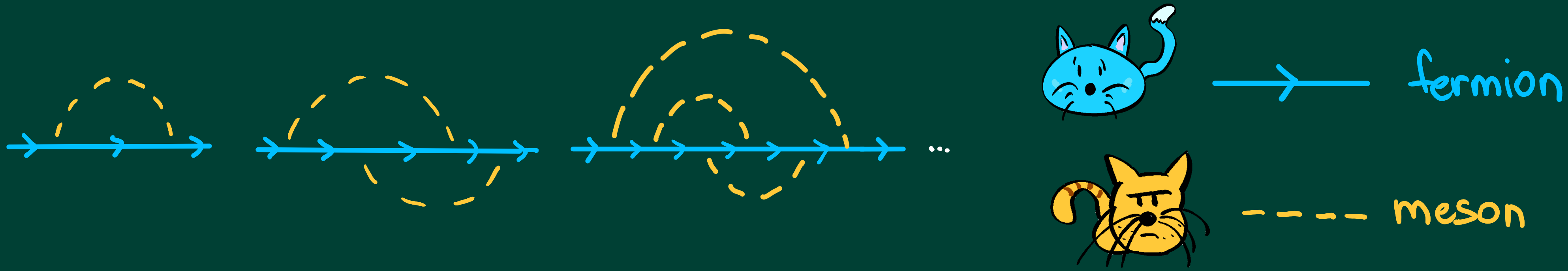
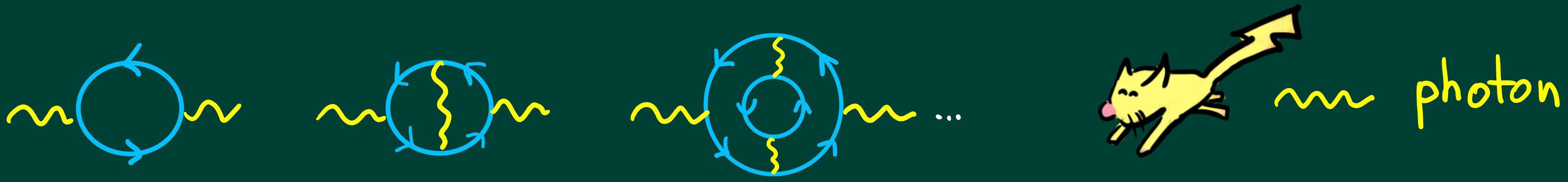
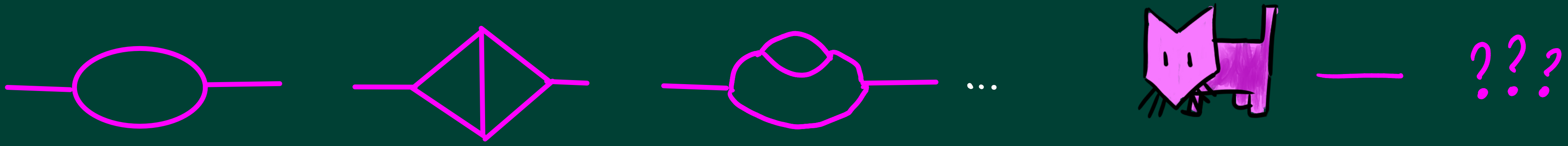


Quantum ElectroDynamics



Scalar ϕ^3 theory



DYSON - SCHWINGER EQUATION

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where $F_k(p)$ is the regularized Feynman integral of the primitive graphs of size k

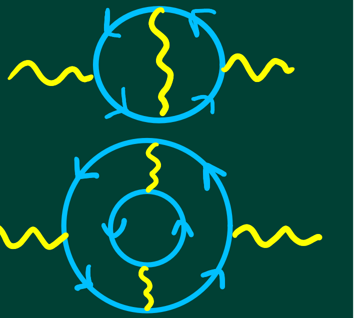
and Δ = insertion growth number

has for solution

$$G(x,L) = \sum_{D \text{ Feynman diagram}} \mathcal{M}_D(L)\, x^{\#\text{vertices}(D)}$$

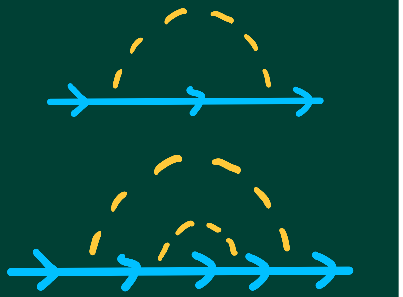
$\Delta=1$

QED



$\Delta=2$

Yukawa



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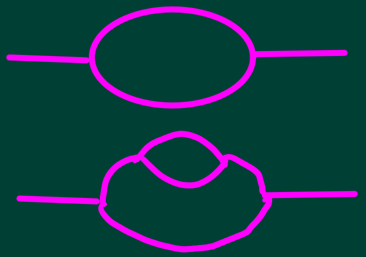

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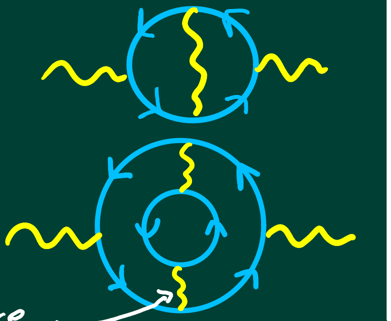

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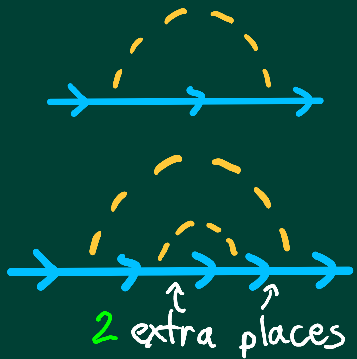

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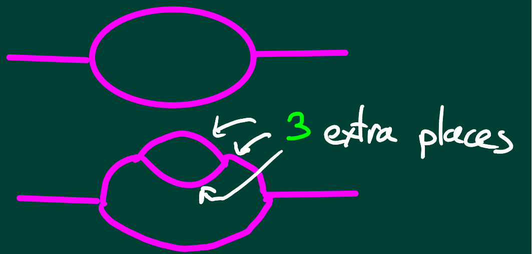

Part II

solving Dyson–Schwinger equations in terms of generating functions

CENTRAL THEOREM

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

$$G(x,L) = 1 - \sum_{k \geqslant 1} x^{k} \, G(x, \partial_{-\rho})^{1-\Delta k} \left(e^{-L\rho} - 1\right) F_k(\rho) \Big|_{\rho=0}$$

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where $F_k(\rho) = \frac{f_{k,0}}{\rho} + f_{k,1} + f_{k,2}\,\rho + f_{k,3}\,\rho^{3} + \dots$

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is the weighted generating function

$$G(x,L) = 1 - \sum_{\substack{C\ \omega_{\Delta}\text{-marked} \\ \text{decorated connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} b_{d(t_1), t_1 - i} \frac{(-L)^{i}}{i!} \right) \prod_{c \text{ non terminal}} b_{d(c),0} \prod_{i=1}^{k-1} b_{d(t_i), t_i - t_{i-1}}\, x^{\|C\|}$$

such that $t_1 < t_2 < \ldots < t_k$ are the positions of the terminal chords for the intersection order

where $$F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2}\,\rho + b_{k,3}\,\rho^{3} + \ldots$$

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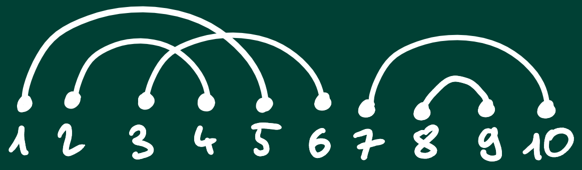
CHORD DIAGRAMS

chord diagram = perfect matching of $\{1,2,\ldots,2n\}$



SMALLEST CHORD DIAGRAMS

$n=1$

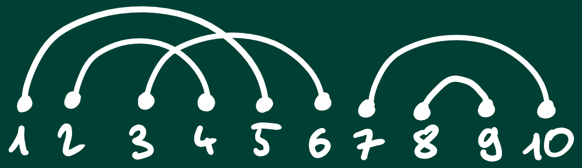
①

$n=2$

③

CHORD DIAGRAMS

chord diagram = perfect matching of $\{1, 2, \ldots, 2n\}$



SMALLEST CHORD DIAGRAMS

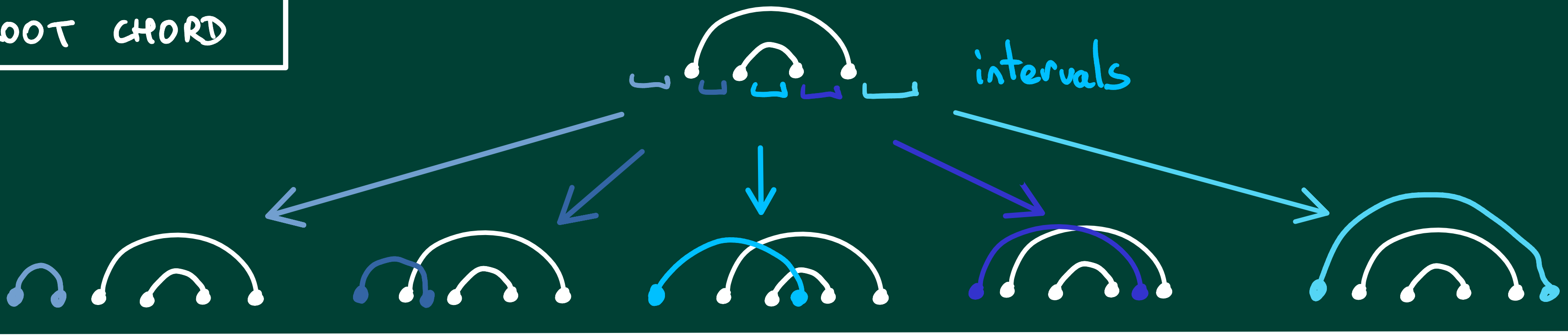
$n = 1$ ①

$n = 2$ ③

number of diagrams with n chords = $(2n-1)!!$

$$= (2n-1) \times (2n-3) \times \ldots \times 3 \times 1$$

HOW TO INSERT A ROOT CHORD



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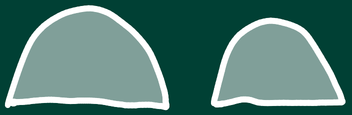
CONNECTED CHORD DIAGRAMS

connected chord diagram =

diagram "in one block" =

diagram <u>not</u> of the form

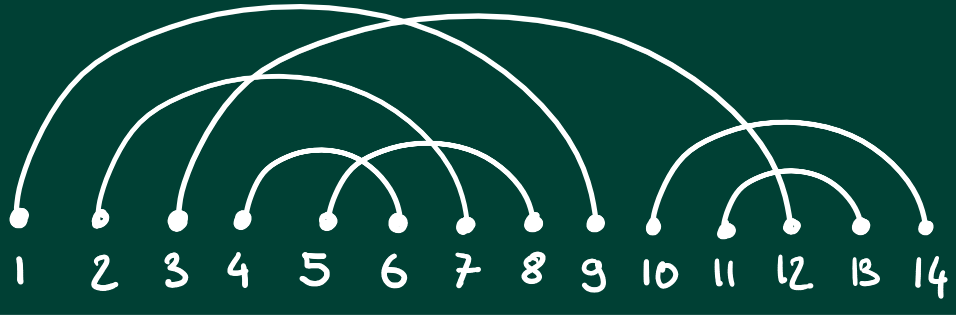


nor

NOT CONNECTED

3 connected components

CONNECTED



SMALLEST CONNECTED CHORD DIAGRAMS

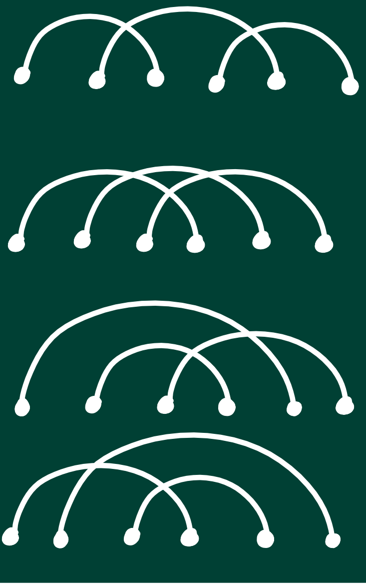
$n = 1$

①

$n = 2$

①

$n = 3$



④

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decorated connected ✓
chord diagram ✓

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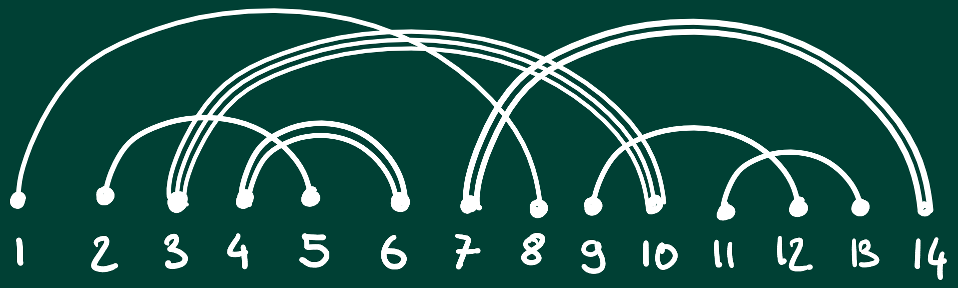
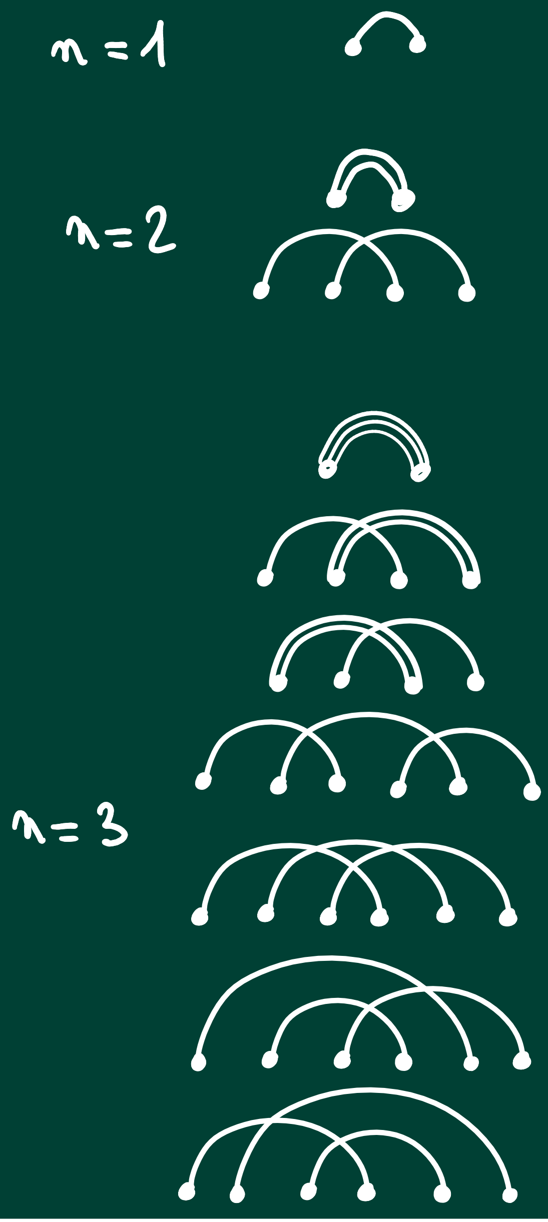
DECORATED CHORD DIAGRAMS

decorated chord diagram =
diagram where each chord
carries a positive integer
= "decoration"



SMALLEST DECORATED CONNECTED CHORD DIAGRAMS



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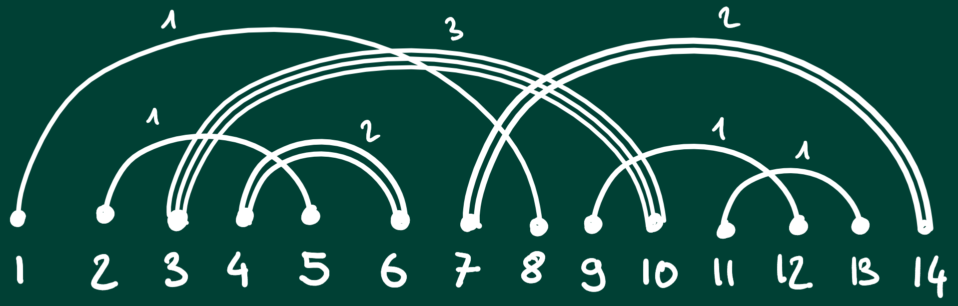

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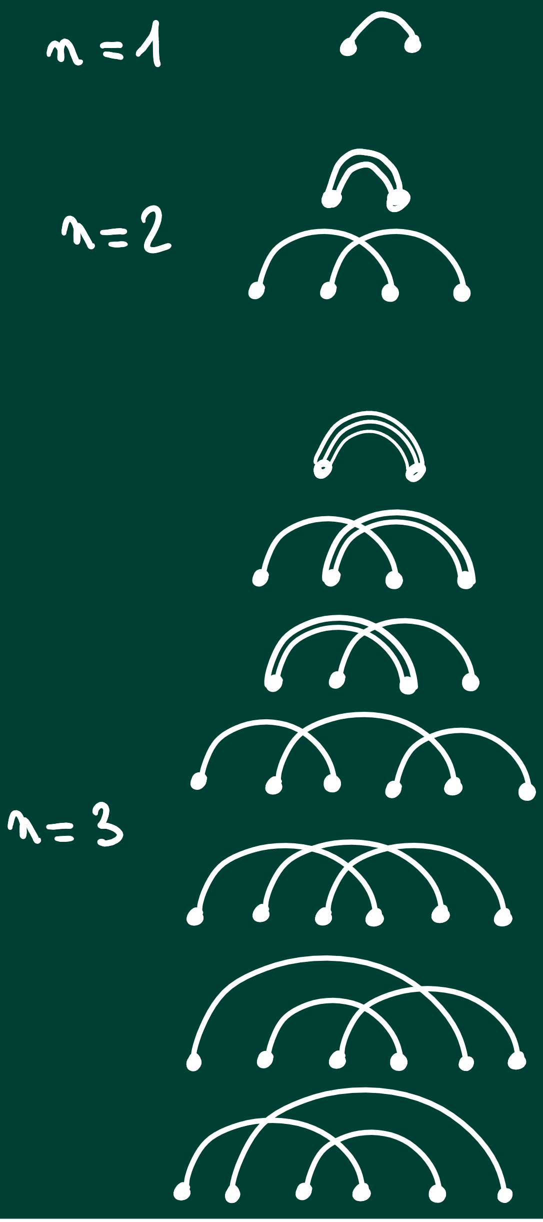

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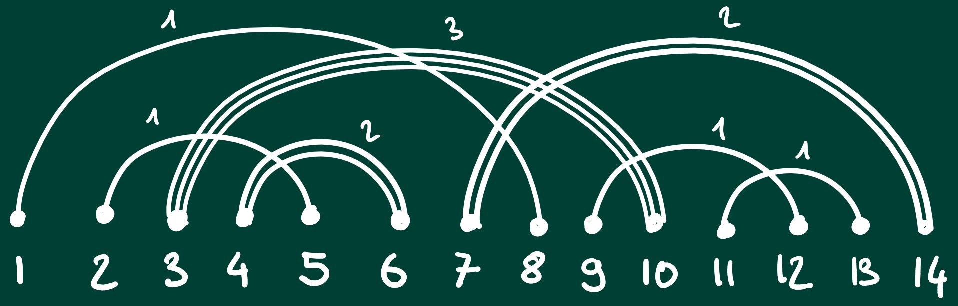
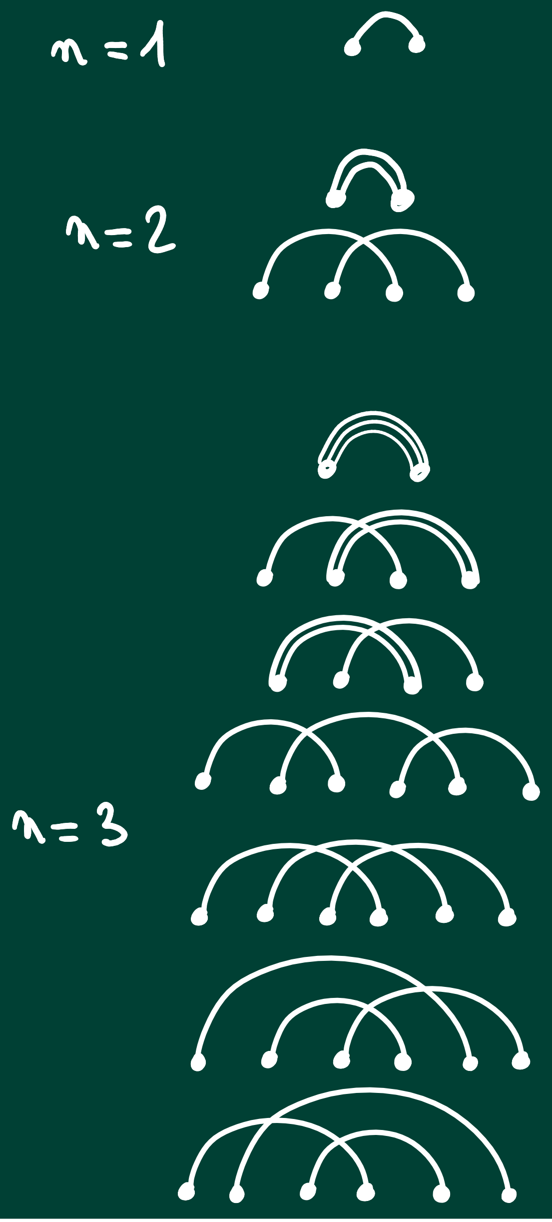


7 chords but size = 11

size of a decorated diagram
= sum of the decorations

SMALLEST DECORATED CONNECTED CHORD DIAGRAMS



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✓ decorated connected ✓
chord diagram ✓

such that $t_1 < t_2 < \ldots < t_k$
are the positions of the terminal chords
for the intersection order

where $F_k(\rho) = \dfrac{\beta_{k,0}}{\rho} + \beta_{k,1} + \beta_{k,2}\,\rho + \beta_{k,3}\,\rho^3 + \ldots$

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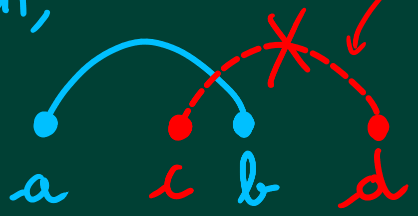
TERMINAL CHORDS

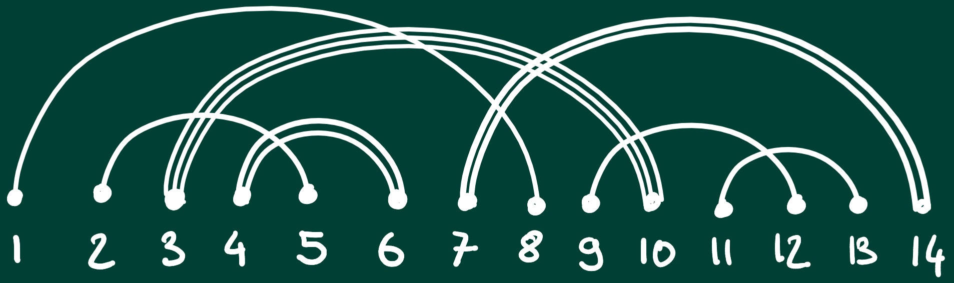
terminal chord =

chord (a, b) such that there is no intersecting chord (c, d) with $b < d$

if terminal,

then forbidden





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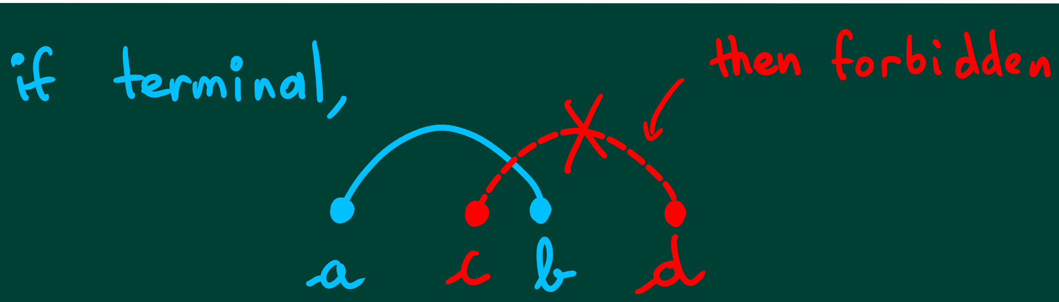
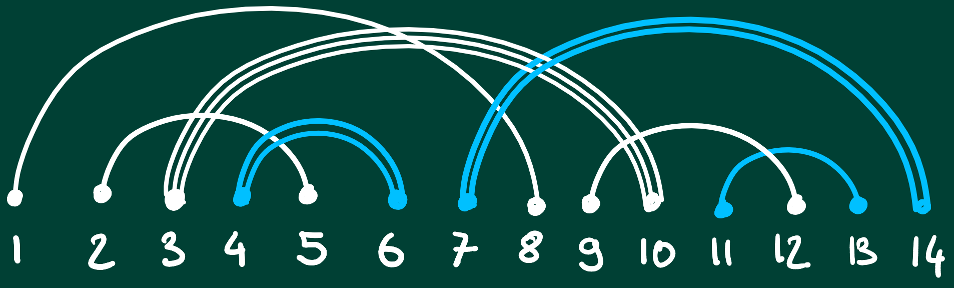
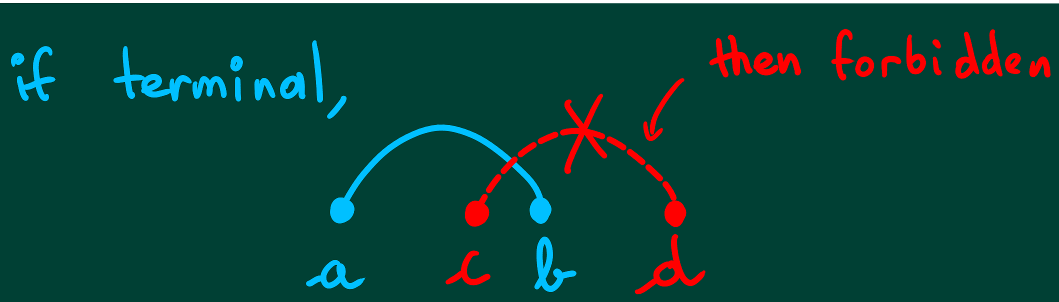




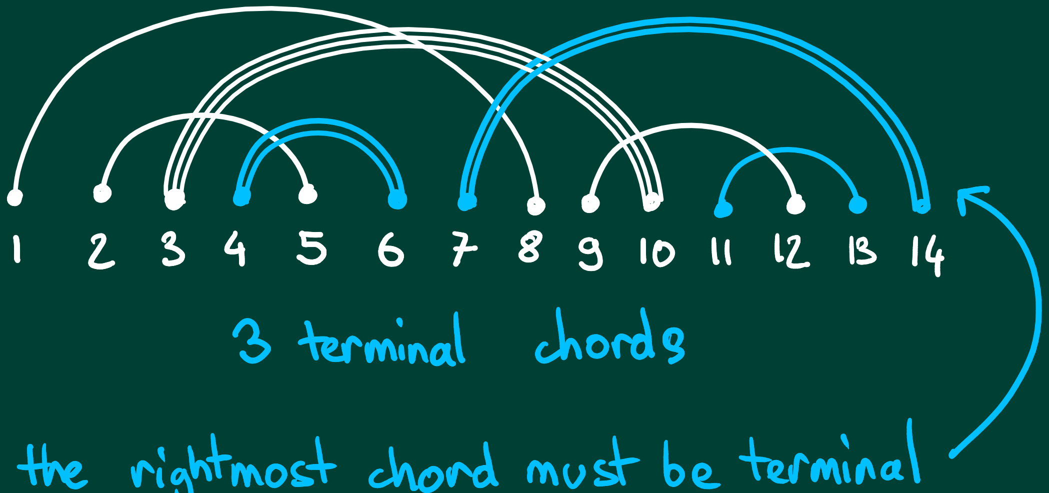
3 terminal chords

TERMINAL CHORDS

terminal chord =
chord (a, b) such that there is
no intersecting chord (c, d)
with $b < d$


if terminal,

then forbidden



3 terminal chords

the rightmost chord must be terminal

CENTRAL THEOREM

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

$$G(x,L) = 1 - \sum_{k\geq 1} x^k \, G(x,\partial_{-\rho})^{1-\Delta k} \left(e^{-L\rho}-1\right) F_k(\rho) \Big|_{\rho=0}$$

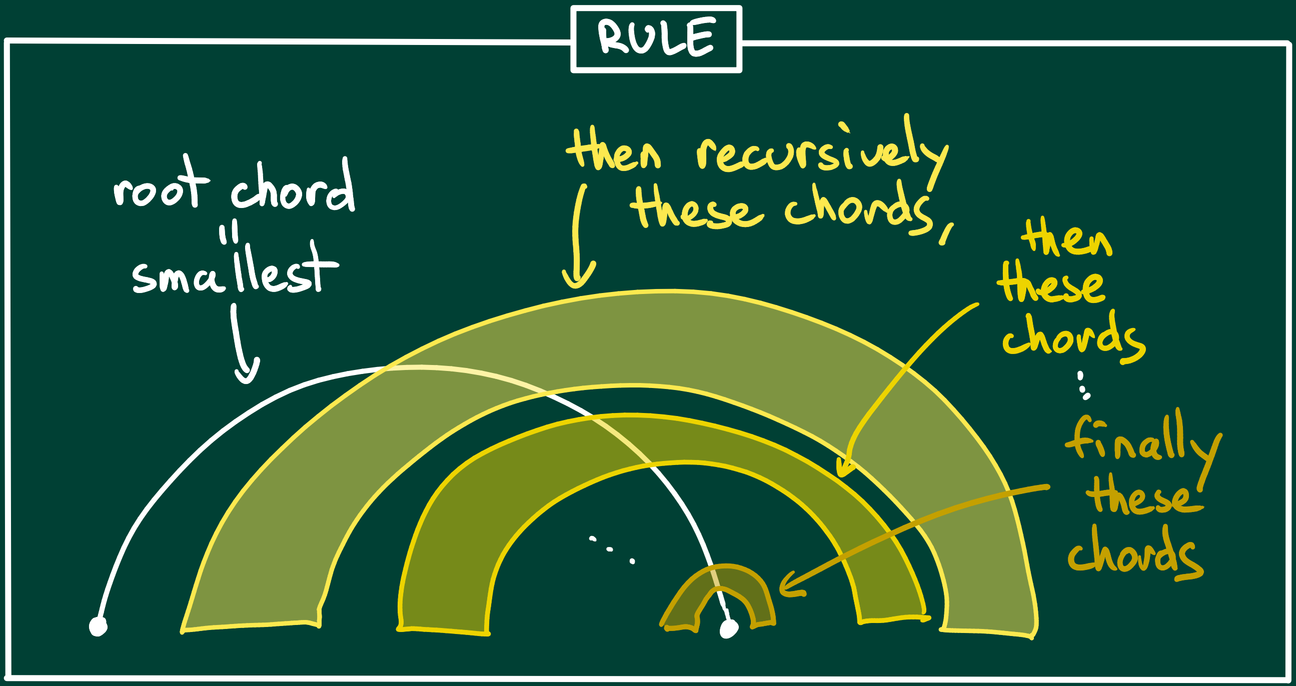
is the weighted generating function

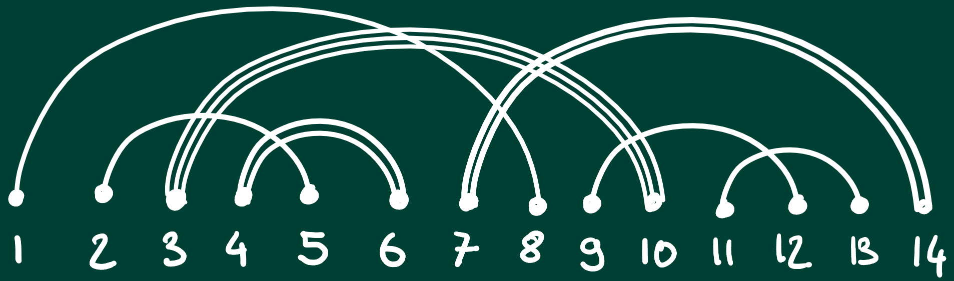
$$G(x,L) = 1 - \sum_{\substack{C\ \omega_\Delta\text{-marked} \\ \checkmark\text{ decorated connected}\checkmark \\ \text{chord diagram}\checkmark}} \left(\sum_{i=1}^{t_1} \beta_{d(t_1),t_1-i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c\text{ non} \\ \text{terminal} \\ \checkmark}} \beta_{d(c),0} \prod_{i=1}^{k-1} \beta_{d(t_i),t_i-t_{i-1}} \, x^{\|C\|}$$

such that $t_1 < t_2 < \ldots < t_k$
are the positions of the terminal chords ✓
for the intersection order

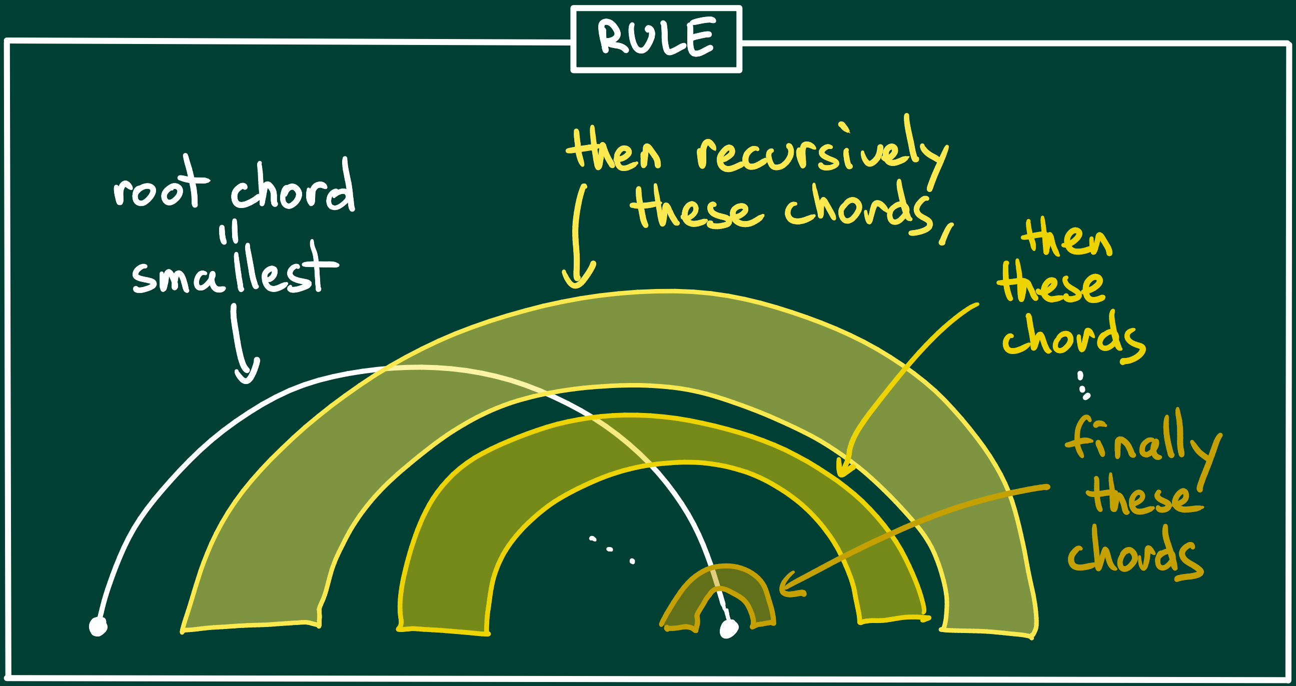
where $F_k(\rho) = \dfrac{\beta_{k,0}}{\rho} + \beta_{k,1} + \beta_{k,2}\,\rho + \beta_{k,3}\,\rho^3 + \ldots$

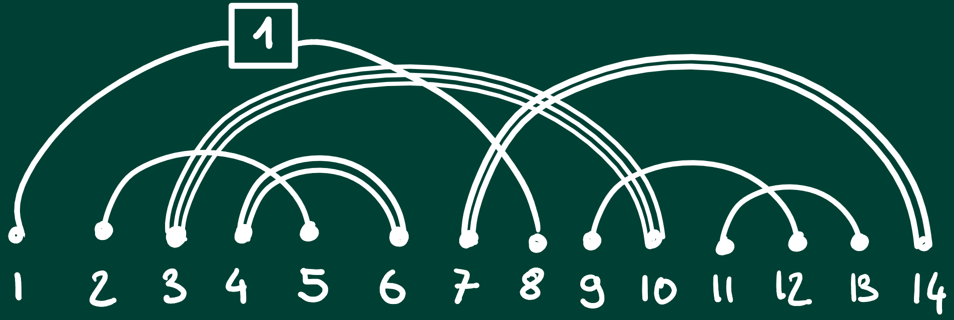
INTERSECTING ORDER



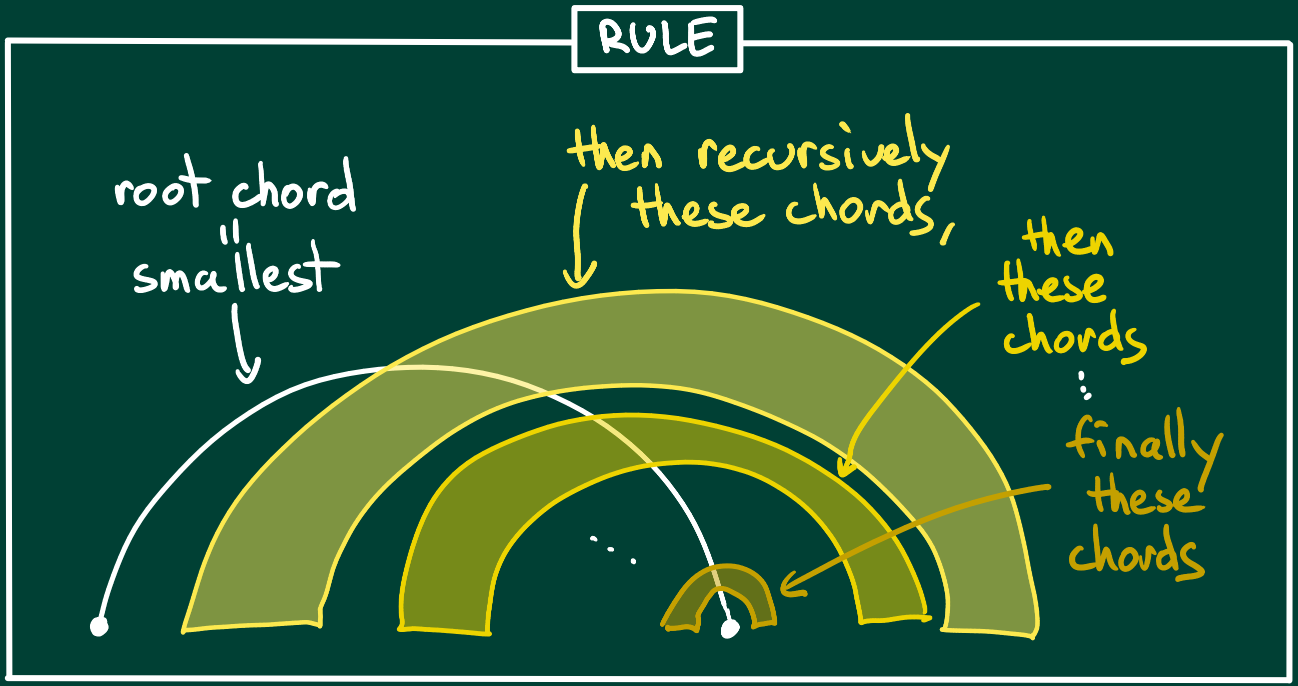


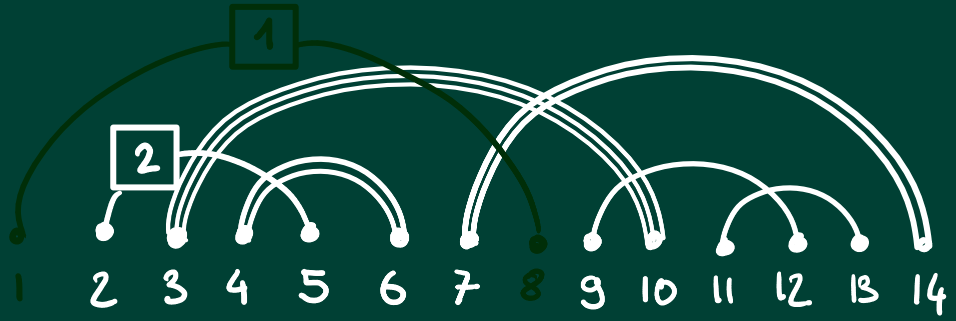
INTERSECTING ORDER





INTERSECTING ORDER





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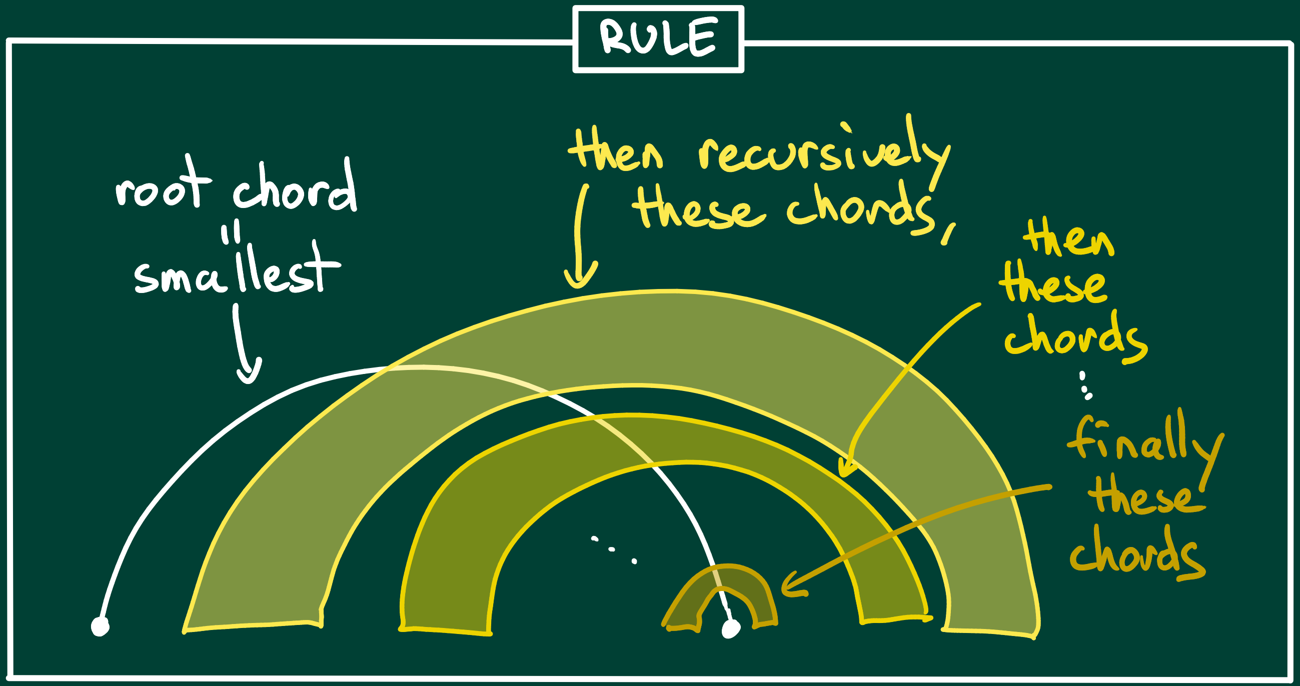
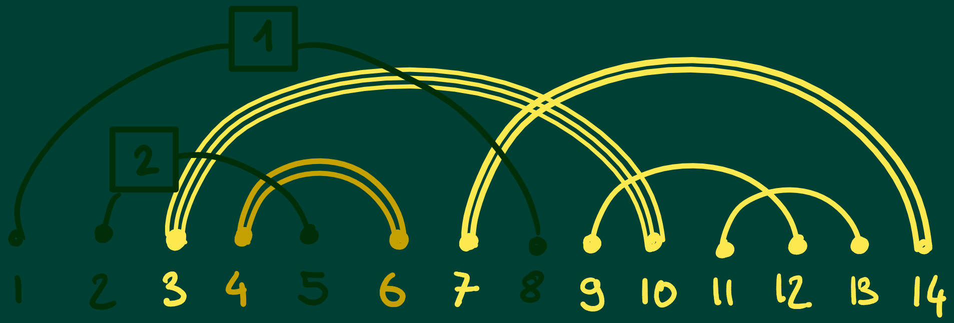
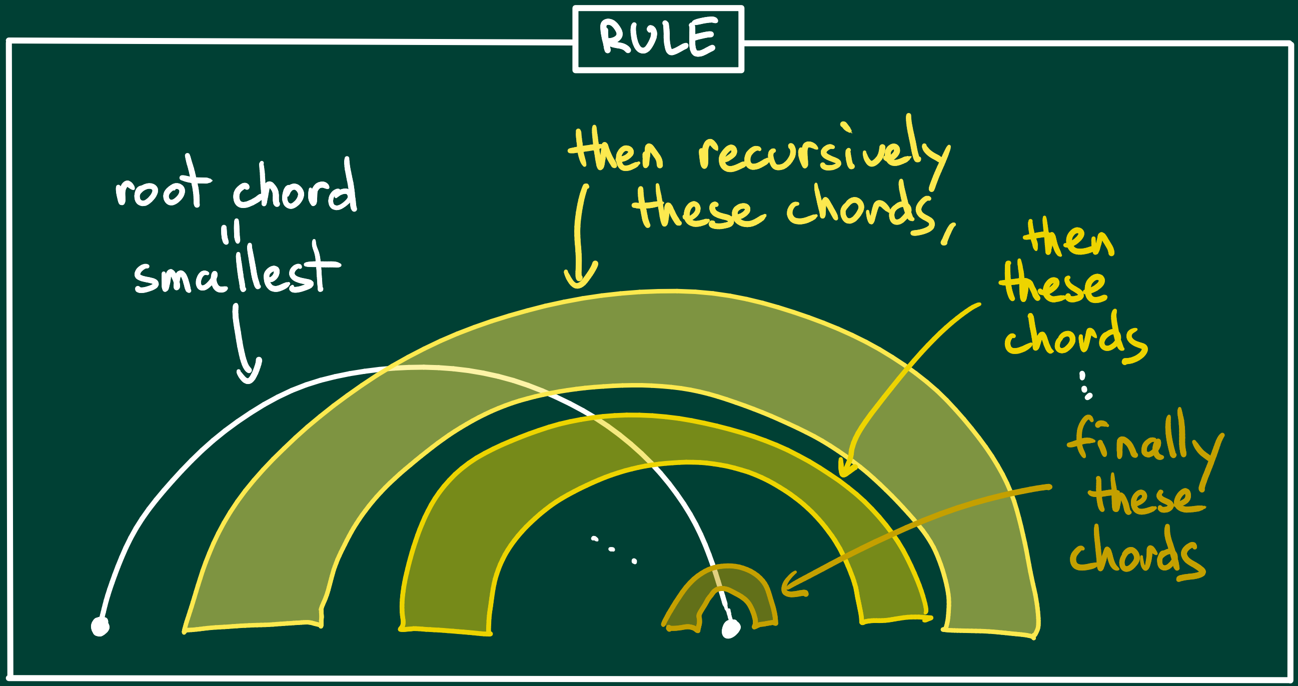
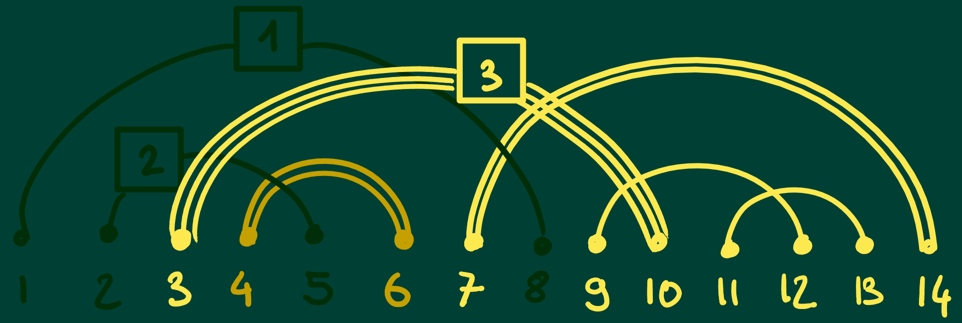




INTERSECTING ORDER





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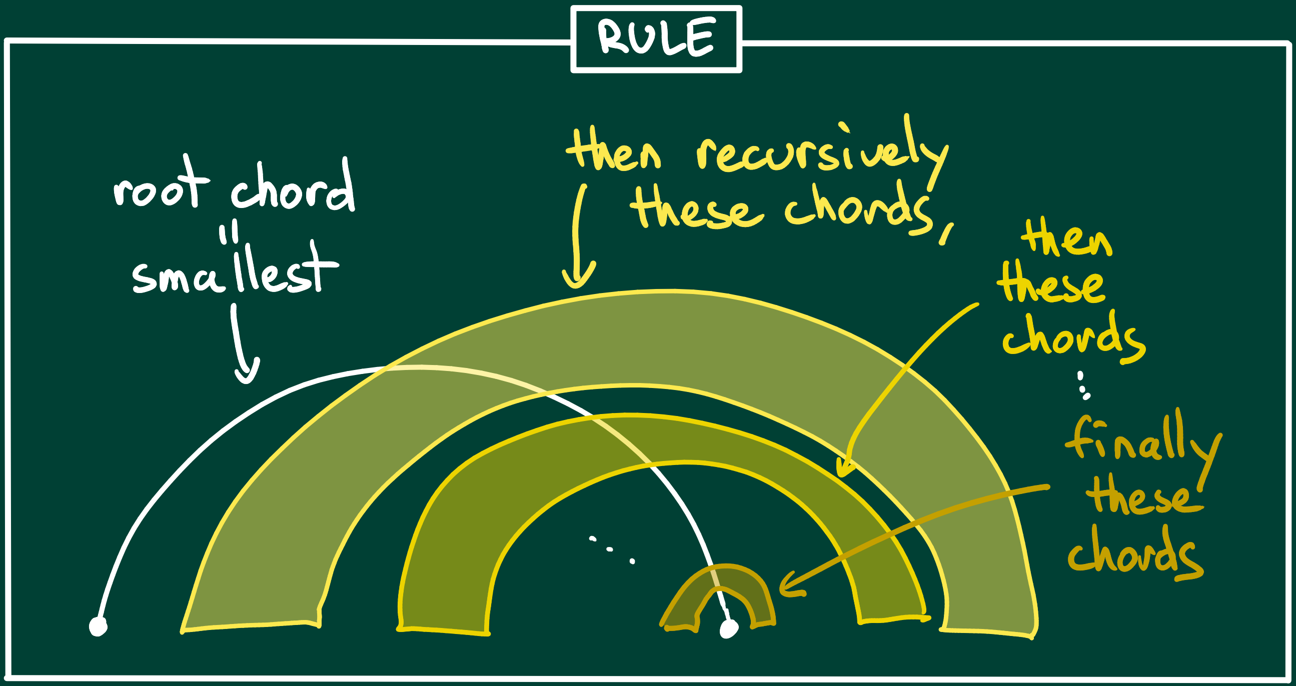
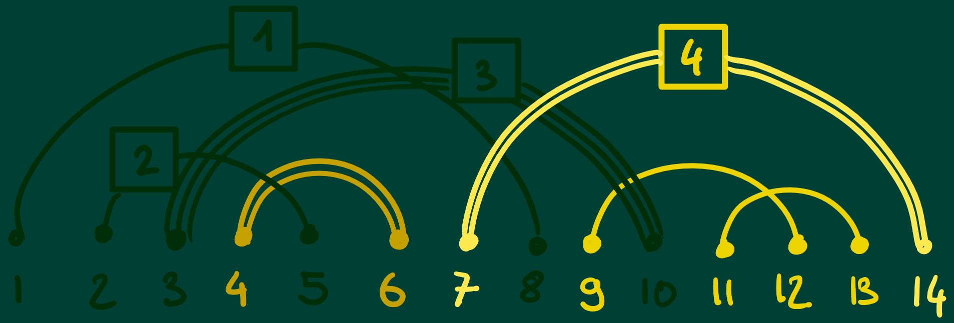
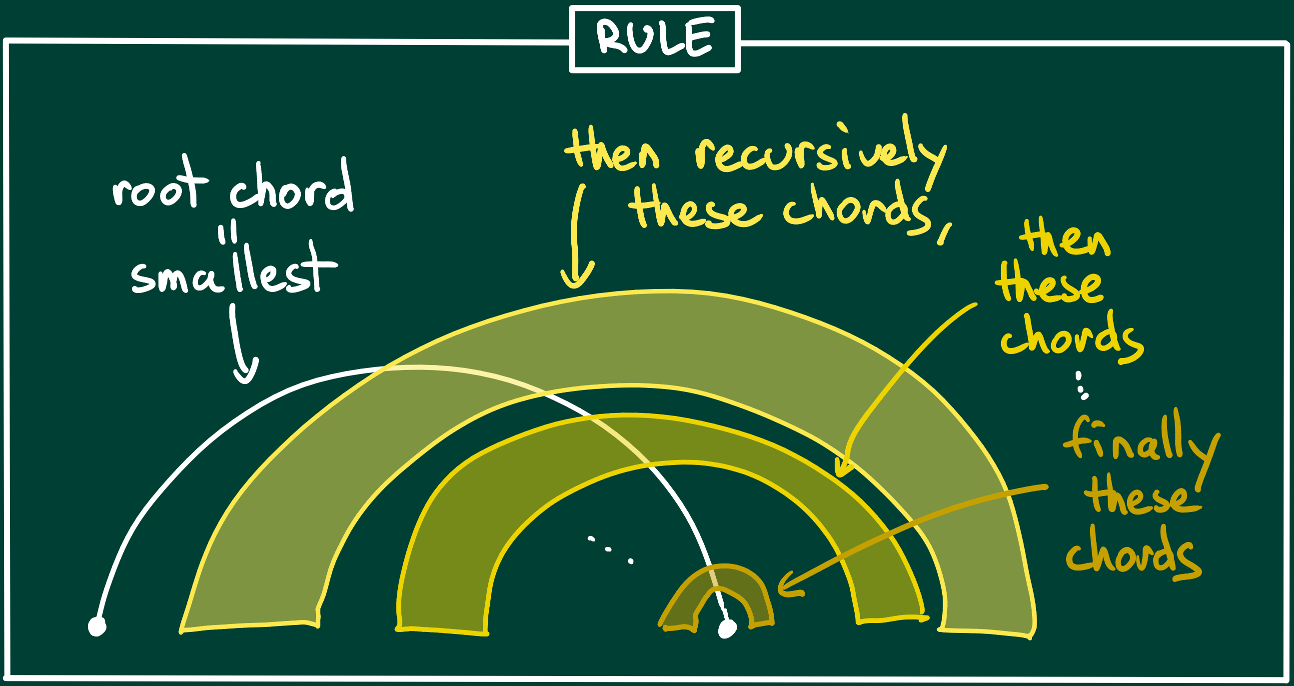
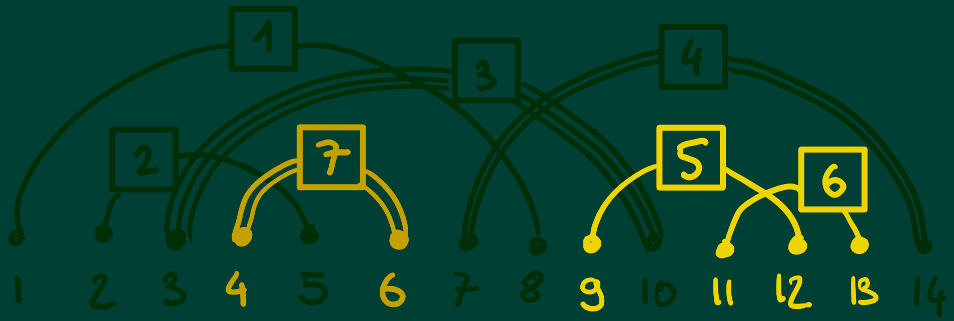




INTERSECTING ORDER





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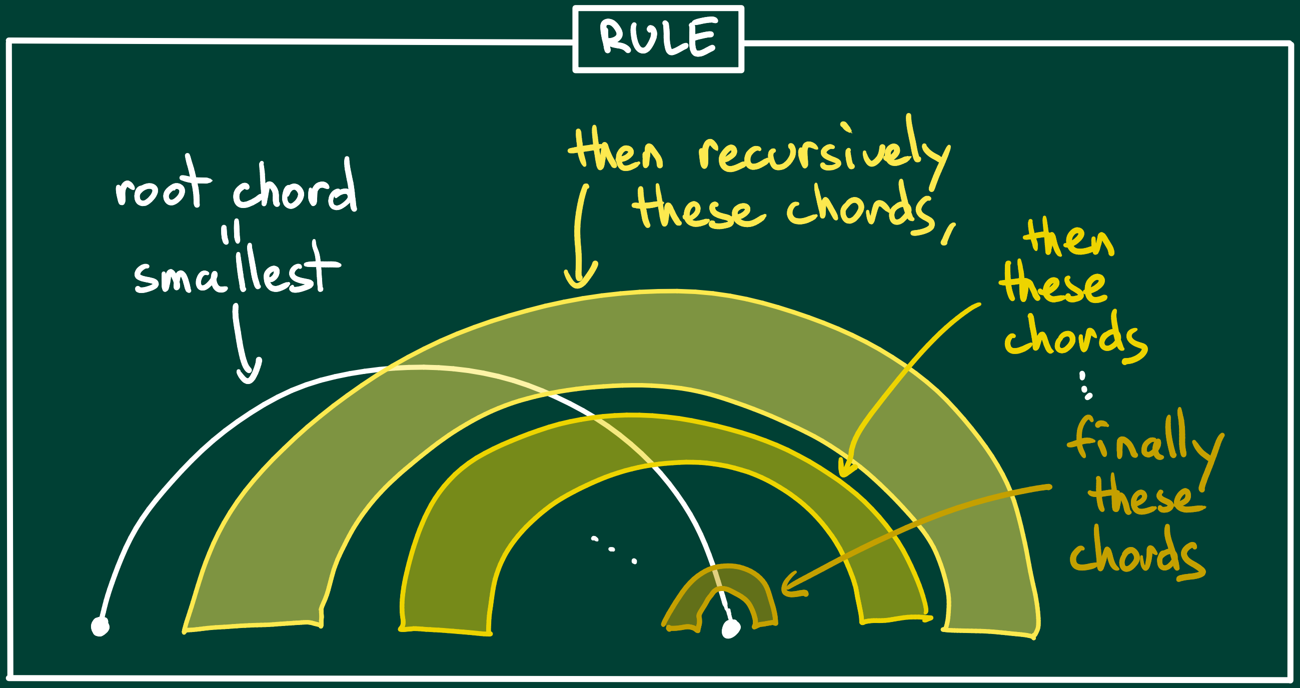
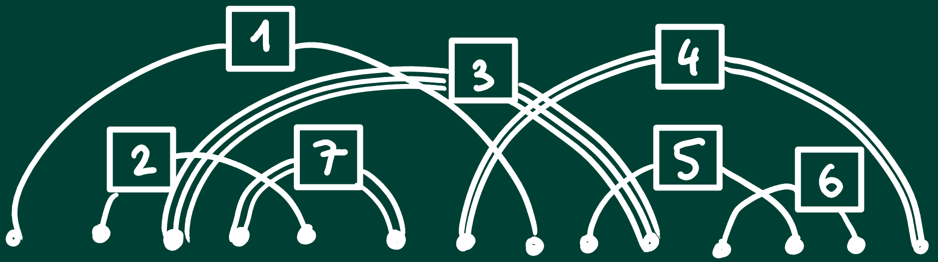
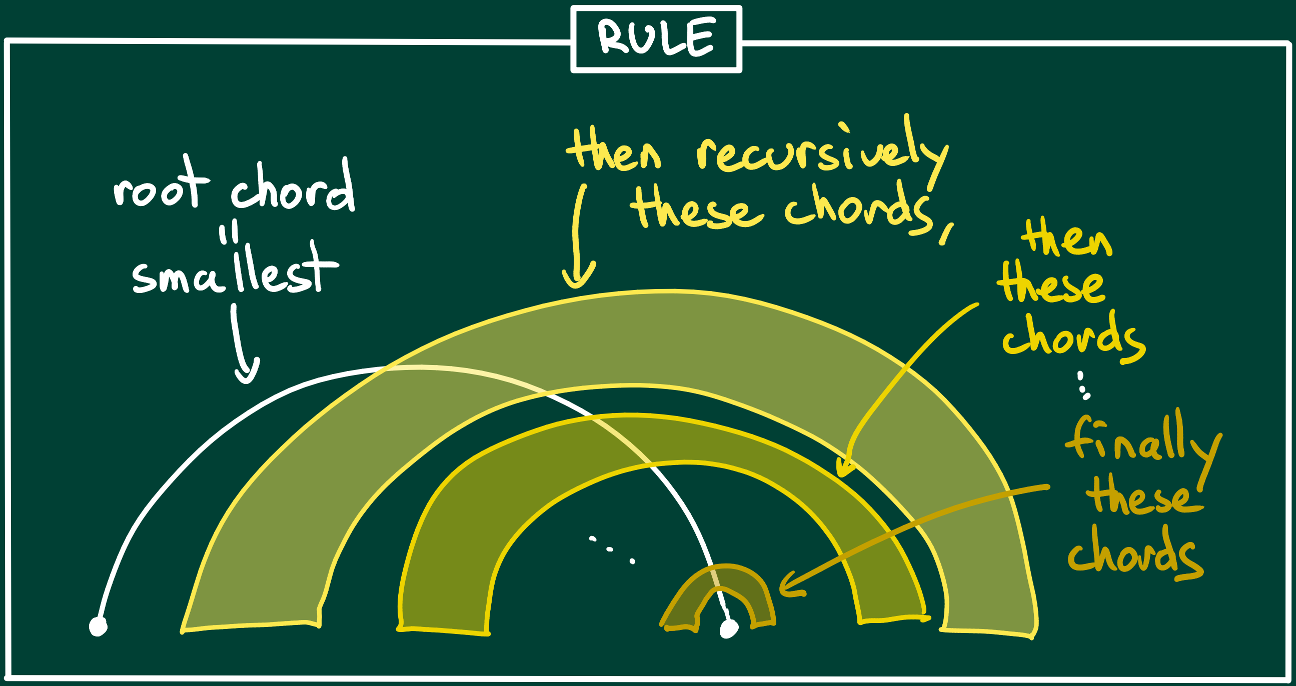
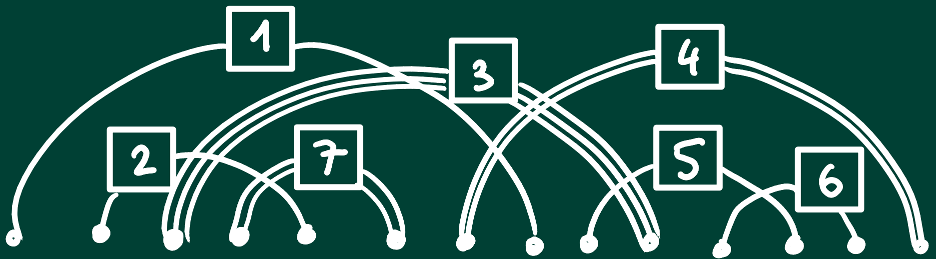




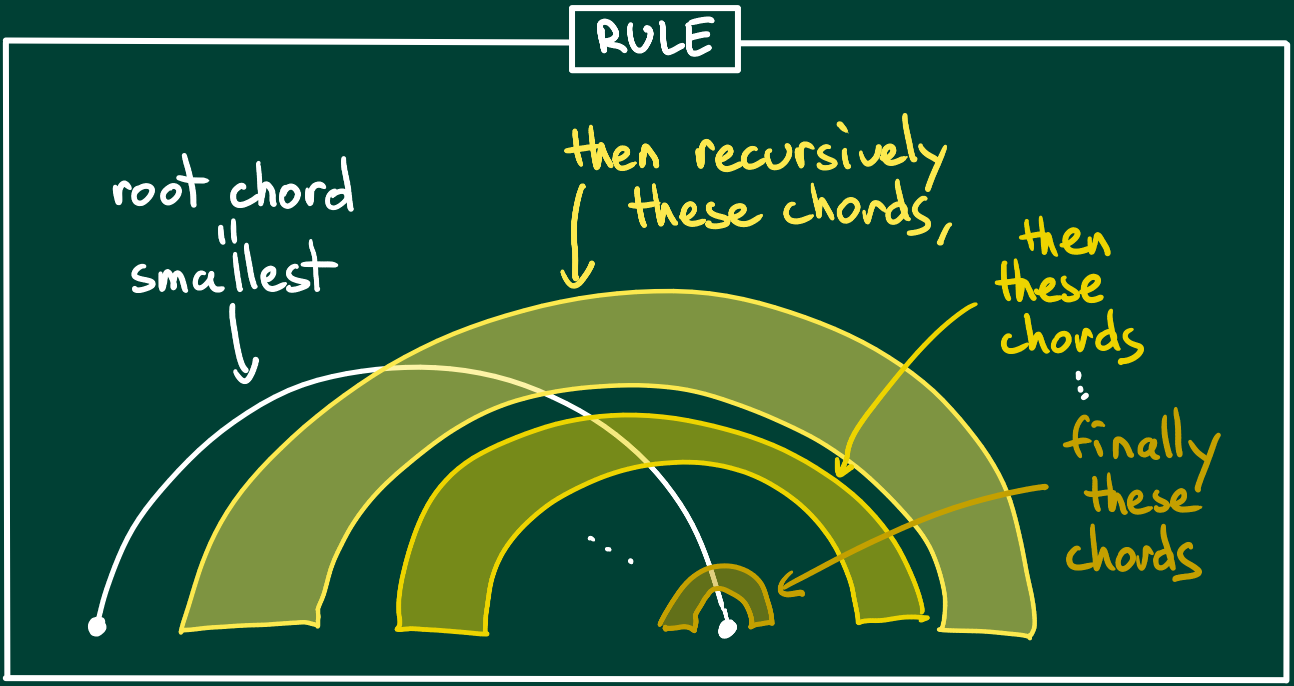
INTERSECTING ORDER

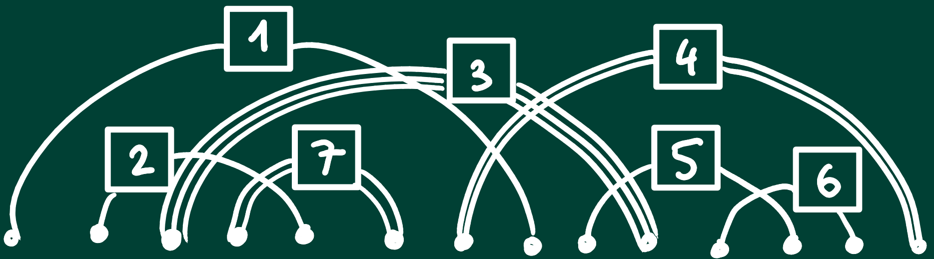




left-to-right
order
≠
intersection
order

INTERSECTING ORDER





left-to-right order ≠ intersection order

From now on, chords will be identified with their positions for the intersecting order

CENTRAL THEOREM

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

$$G(x,L) = 1 - \sum_{k \geq 1} x^k \, G(x, \partial_{-\rho})^{1-\Delta k} \left(e^{-L\rho} - 1\right) F_k(\rho) \Big|_{\rho=0}$$

is the weighted generating function

$$G(x,L) = 1 - \sum_{\substack{C\ \omega_\Delta\text{-marked} \\ \text{decorated connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} b_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal}}} b_{d(c),0} \prod_{i=1}^{k-1} b_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

✓ decorated connected ✓ chord diagram ✓

such that $t_1 < t_2 < \dots < t_k$ are the positions of the terminal chords ✓ for the intersection order ✓

non terminal ✓

where

$$F_k(\rho) = \frac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2}\,\rho + b_{k,3}\,\rho^3 + \dots$$

CENTRAL THEOREM

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$$G(x,L) = 1 - \sum_{k \geq 1} x^k \, G(x, \partial_{-p})^{1-\Delta k} \left(e^{-Lp} - 1\right) F_k(p) \Big|_{p=0}$$

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$$G(x,L) = 1 - \sum_{\substack{C \ \omega_\Delta\text{-marked ?} \\ \checkmark \text{ decorated connected } \checkmark \\ \text{chord diagram } \checkmark}} \left(\sum_{i=1}^{t_1} \beta_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal} \\ \checkmark}} \beta_{d(c),0} \prod_{i=1}^{k-1} \beta_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

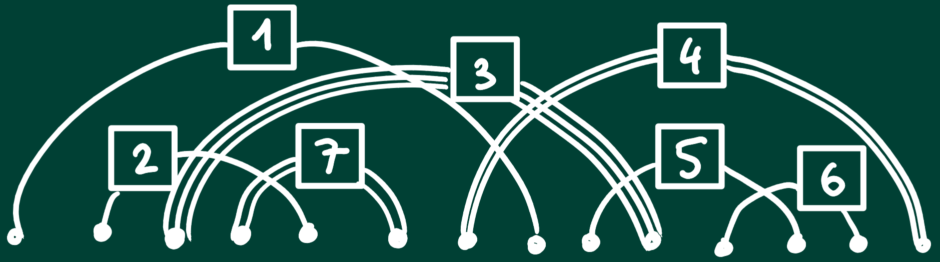
such that $t_1 < t_2 < \ldots < t_k$
are the positions of the terminal chords ✓
for the intersection order ✓

where $F_k(p) = \dfrac{\beta_{k,0}}{p} + \beta_{k,1} + \beta_{k,2}\, p + \beta_{k,3}\, p^3 + \ldots$

INTERVAL COVERING OF CHORDS

ALGO

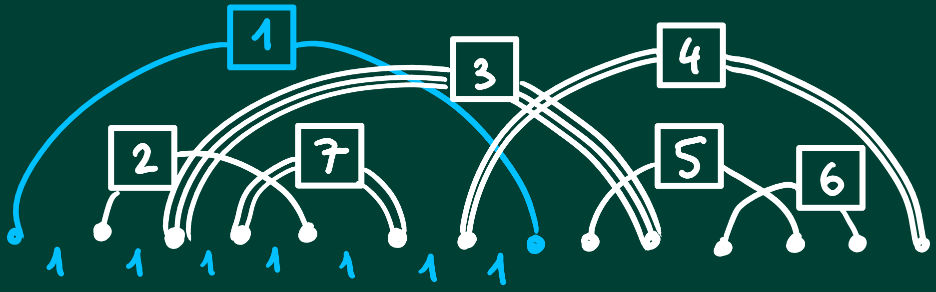
For each chord i (in the intersection order)
Label each interval below i by i
(erase the previous label if needed)



INTERVAL COVERING OF CHORDS

ALGO

```
For each chord i (in the intersection order)
    Label each interval below i by i
    (erase the previous label if needed)
```



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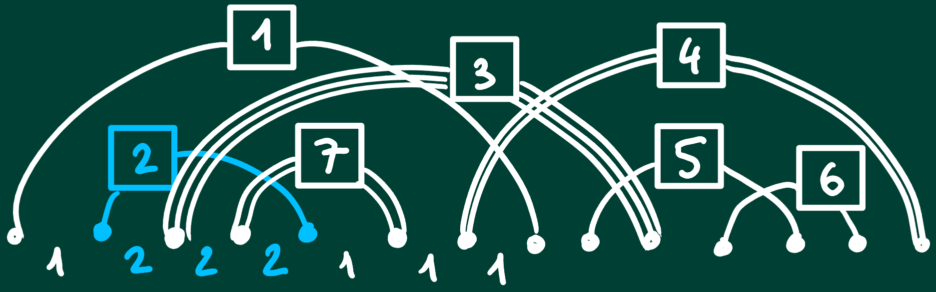

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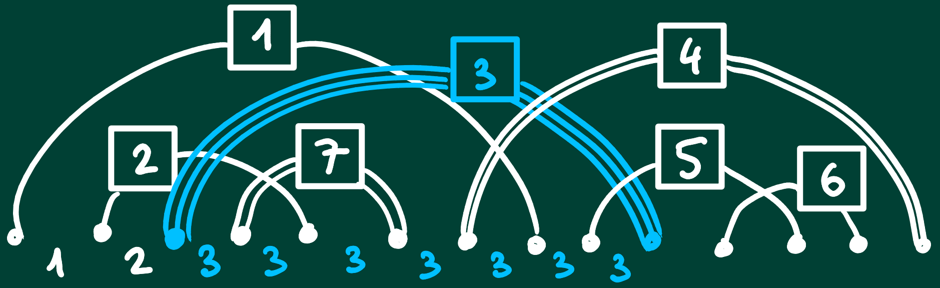

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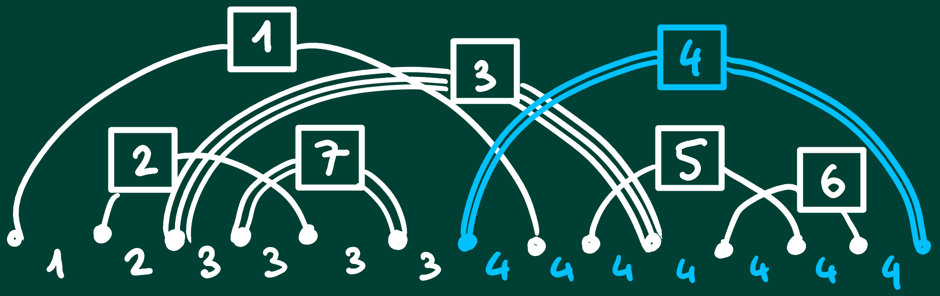

INTERVAL COVERING OF CHORDS

ALGO

```
For each chord $i$ (in the intersection order)
    Label each interval below $i$ by $i$
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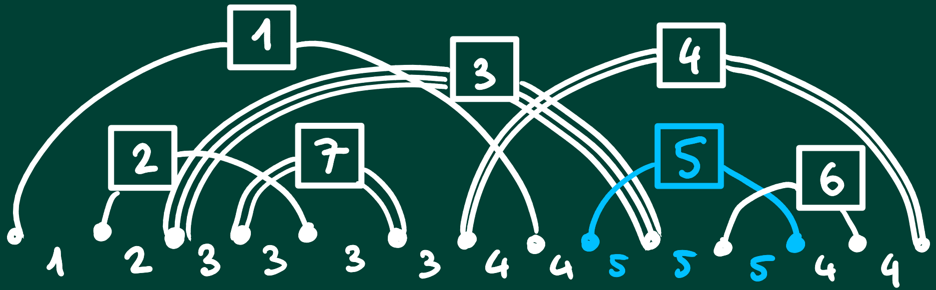
```



INTERVAL COVERING OF CHORDS

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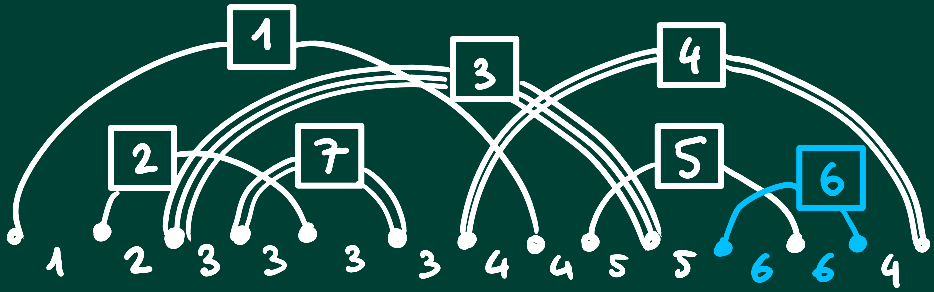

INTERVAL COVERING OF CHORDS

ALGO

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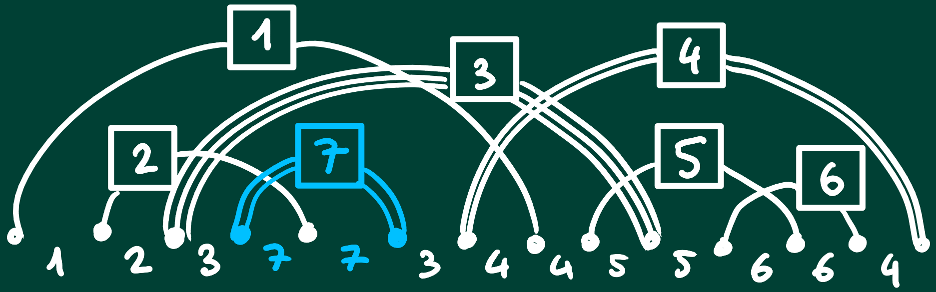
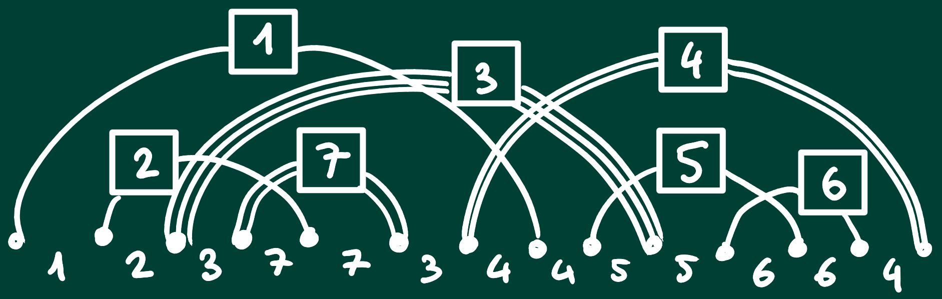
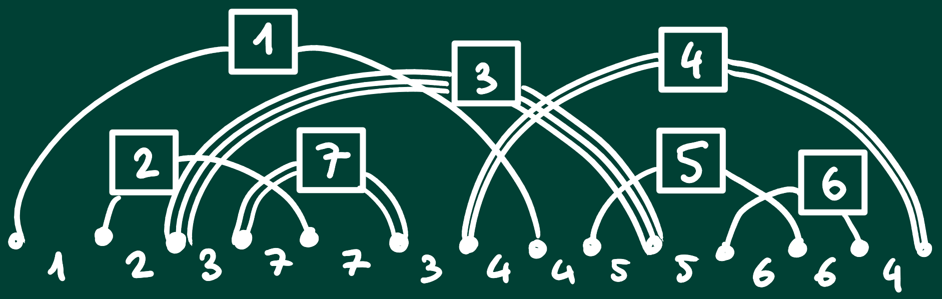
```



ω_D- MARKED DIAGRAM



ω_Δ- MARKED DIAGRAM



DEFINITION

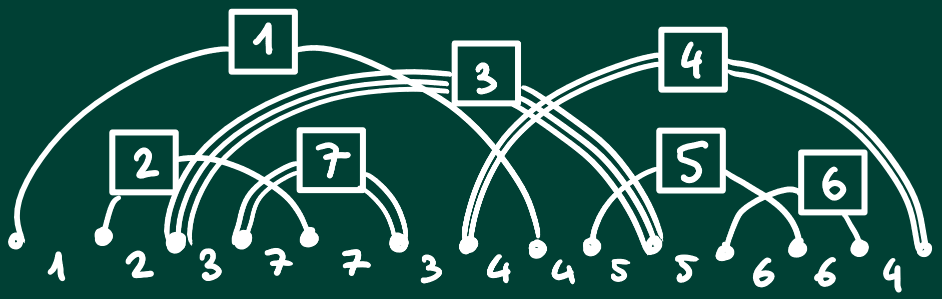
For any integer $\Delta \geq 2$

ω_Δ-marked diagram = diagram where

- each interval may contain marks (= crosses), horizontally arranged
- for each chord c, the intervals covered by c must contain $\Delta \times d(c) - 2$ marks in total.

e.g. for $\Delta = 2$

ω_Δ - MARKED DIAGRAM



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ω_Δ-marked diagram = diagram where

- each interval may contain marks (= crosses), horizontally arranged
- for each chord c, the intervals covered by c must contain $\Delta \times d(c) - 2$ marks in total.

e.g for $\Delta = 2$

($\Delta = 1$ is a bit more complex)

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

$$G(x,L) = 1 - \sum_{k \geq 1} x^{k} \, G(x, \partial_{-\rho})^{1-\Delta k} \left(e^{-L\rho} - 1\right) F_{k}(\rho) \Big|_{\rho=0}$$

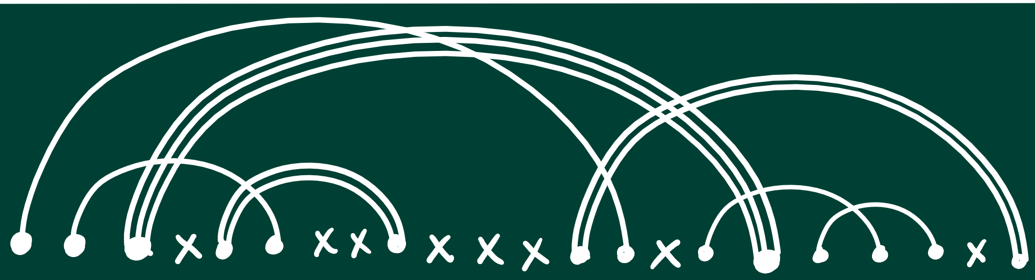
is the weighted generating function

$$G(x,L) = 1 - \sum_{\substack{C \ \omega_{\Delta}\text{-marked} \\ \text{decorated connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} b_{d(t_1), t_1 - i} \frac{(-L)^{i}}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal}}} b_{d(c),0} \prod_{i=1}^{k-1} b_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

such that $t_1 < t_2 < \dots < t_k$ are the positions of the terminal chords for the intersection order

where $F_k(\rho) = \dfrac{b_{k,0}}{\rho} + b_{k,1} + b_{k,2}\,\rho + b_{k,3}\,\rho^{3} + \dots$

e.g:



THEOREM [Hihn Yeats] [Courtiel Yeats]

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$$G(x,L) = 1 - \sum_{k \geqslant 1} x^k G(x, \partial_{-\rho})^{1-\Delta k} (e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$

is the weighted generating function

$$G(x,L) = 1 - \sum_{\substack{C\ \omega_\Delta\text{-marked} \\ \text{decorated connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} f_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal}}} f_{d(c),0} \prod_{i=1}^{k-1} f_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

such that $t_1 < t_2 < \ldots < t_k$
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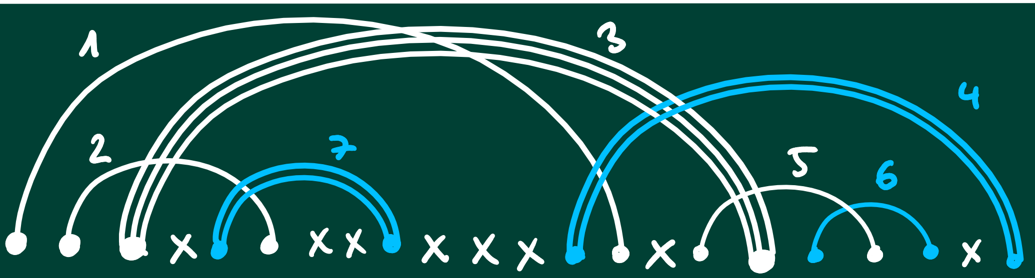
e.g:



$t_1 = 4 \quad d(t_1) = 2$

$t_2 = 6 \quad d(t_2) = 1$

$t_3 = 7 \quad d(t_3) = 2$

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

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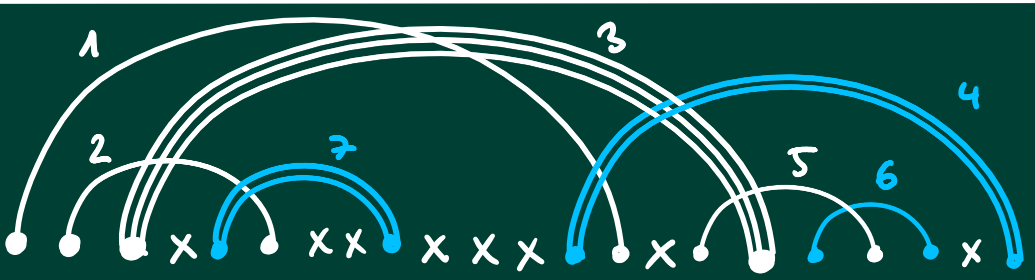
such that $t_1 < t_2 < \dots < t_k$
are the positions of the terminal chords
for the intersection order

where $F_k(\rho) = \dfrac{f_{k,0}}{\rho} + f_{k,1} + f_{k,2}\,\rho + f_{k,3}\,\rho^3 + \dots$

e.g.:

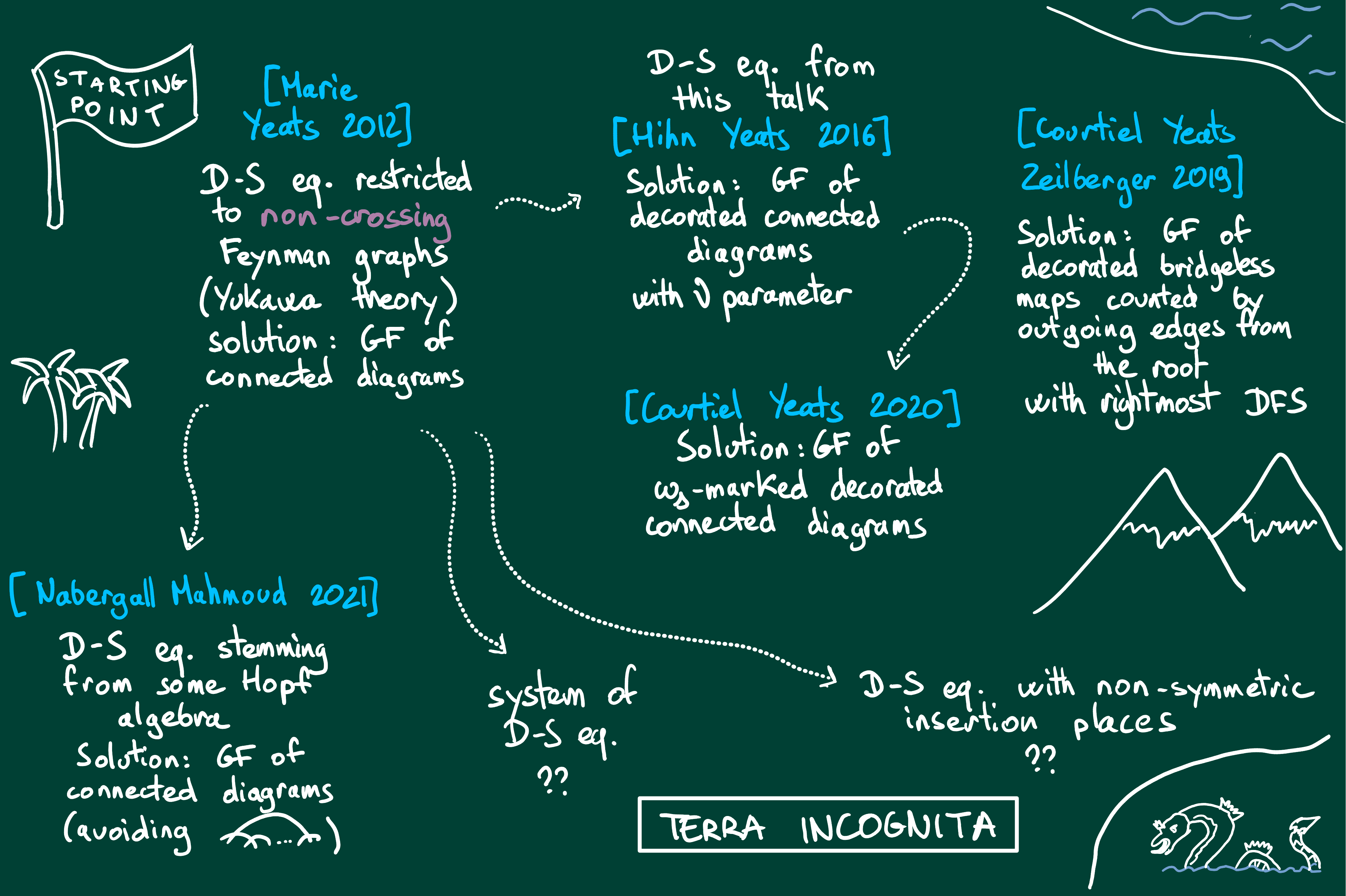
Weight of



$t_1 = 4 \quad d(t_1) = 2$
$t_2 = 6 \quad d(t_2) = 1$
$t_3 = 7 \quad d(t_3) = 2$

$$= \left(f_{2,0} \frac{L^4}{4!} - f_{2,1} \frac{L^3}{3!} + f_{2,2} \frac{L^2}{2} - f_{2,3} L \right) f_{1,0}^{\,3} \, f_{3,0} \, f_{1,2} \, f_{2,1} \, x^{11}$$

ROADMAP OF THE EXPLORATION OF "SOLVING DYSON-SCHWINGER EQUATIONS AS GENERATING FUNCTIONS"



[Marie Yeats 2012]
D-S eq. restricted to non-crossing Feynman graphs (Yukawa theory)
solution: GF of connected diagrams

D-S eq. from this talk
[Hihn Yeats 2016]
Solution: GF of decorated connected diagrams
with ϑ parameter

[Courtiel Yeats Zeilberger 2019]
Solution: GF of decorated bridgeless maps counted by outgoing edges from the root
with rightmost DFS

[Courtiel Yeats 2020]
Solution: GF of ω_s-marked decorated connected diagrams

[Nabergall Mahmoud 2021]
D-S eq. stemming from some Hopf algebra
Solution: GF of connected diagrams (avoiding ⌒⌒…⌒)

system of D-S eq.
??

D-S eq. with non-symmetric insertion places
??

TERRA INCOGNITA

CENTRAL THEOREM

THEOREM [Hihn Yeats] [Courtiel Yeats]

The solution of

$$G(x,L) = 1 - \sum_{k \geq 1} x^k \, G(x, \partial_{-\rho})^{1-\Delta k} \left(e^{-L\rho} - 1\right) F_k(\rho) \Big|_{\rho=0}$$

is the weighted generating function

$$G(x,L) = 1 - \sum_{\substack{C \ \omega_\Delta\text{-marked} \\ \text{decorated connected} \\ \text{chord diagram}}} \left(\sum_{i=1}^{t_1} \beta_{d(t_1),t_1-i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal}}} \beta_{d(c),0} \prod_{i=1}^{k-1} \beta_{d(t_i),t_i-t_{i-1}} \, x^{\|C\|}$$

such that $t_1 < t_2 < \dots < t_k$ are the positions of the terminal chords for the intersection order

where $$F_k(\rho) = \frac{\beta_{k,0}}{\rho} + \beta_{k,1} + \beta_{k,2}\,\rho + \beta_{k,3}\,\rho^3 + \dots$$

What to do with this (awful?) formula?

Part III

Leading-log expansions

LEADING-LOG EXPANSIONS

$$G(x,L) = 1 - \sum_{\substack{C\ \omega_D\text{-marked} \\ \text{decorated connected chord diagram} \\ \text{such that } t_1 < t_2 < \dots < t_k \\ \text{are the positions of the terminal chords}}} \left(\sum_{i=1}^{t_1} f_{d(t_1),t_1-i} \frac{(-L)^i}{i!} \right) \prod_{\substack{c \text{ non} \\ \text{terminal}}} f_{d(c),0} \prod_{i=1}^{k-1} f_{d(t_i),t_i-t_{i-1}} \, x^{\|C\|}$$

is of the form $G(x,L) - 1 = g_{1,1}\, x L$

$$+ g_{2,1}\, x^2 L + g_{2,2}\, x^2 L^2$$

$$+ g_{3,1}\, x^3 L + g_{3,2}\, x^3 L^2 + g_{3,3}\, x^3 L^3$$

$$+ \dots$$

LEADING-LOG EXPANSIONS

$$G(x,L) = 1 - \sum_{\substack{C\ \omega_0\text{-marked} \\ \text{decorated connected chord diagram} \\ \text{such that } t_1 < t_2 < \ldots < t_k \\ \text{are the positions of the terminal chords}}} \left(\sum_{i=1}^{t_1} f_{d(t_1), t_1 - i} \frac{(-L)^i}{i!} \right) \prod_{c \text{ non terminal}} f_{d(c),0} \prod_{i=1}^{k-1} f_{d(t_i), t_i - t_{i-1}} \, x^{\|C\|}$$

is of the form $G(x,L) - 1 = g_{1,1}\, xL$

$$+ g_{2,1}\, x^2 L + g_{2,2}\, x^2 L^2$$

$$+ g_{3,1}\, x^3 L + g_{3,2}\, x^3 L^2 + g_{3,3}\, x^3 L^3$$

$$+ \ldots$$

We can write $G(x,L) = 1 + \sum_{k \geq 1} H_k(xL) \times x^k$

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$$\begin{aligned} G(x,L) - 1 = {} & g_{1,1}\, xL \\ & + g_{2,1}\, x^2 L + g_{2,2}\, x^2 L^2 \\ & + g_{3,1}\, x^3 L + g_{3,2}\, x^3 L^2 + g_{3,3}\, x^3 L^3 \\ & + \dots \end{aligned}$$

H_0

We can write $G(x,L) = 1 + \sum_{k \geq 1} H_k(xL) \times x^k$

DEFINITION

$H_0(z)$ = leading-log expansion

LEADING-LOG EXPANSIONS

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H_1 H_0

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DEFINITION

$H_0(z)$ = leading-log expansion

$H_1(z)$ = next-to leading-log expansion

LEADING-LOG EXPANSIONS

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is of the form

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H_2 H_1 H_0

We can write $G(x,L) = 1 + \sum_{k \geq 1} H_k(xL) \times x^k$

DEFINITION

[Krüger Kreimer 2015]

$H_0(z)$ = leading-log expansion

$H_1(z)$ = next-to leading-log expansion

$H_k(z) = \underbrace{\text{next-to next-to} \dots \text{next-to}}_{k \text{ times}}$ leading-log expansion.

LEADING LOG EXPANSIONS IN TERMS OF CHORD DIAGRAMS

$$\boxed{\text{Leading\_log\_expansion}(z) = -F_{\text{one\_term\_chord}}(-\beta_{1,0}\, z)}$$

where $F_{\text{one\_term\_chord}}(z)$ = exponential generating function of $\omega_{\triangle}$-marked connected diagrams with only one terminal chord (all decorations = 1)

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$$\boxed{\begin{gathered}\text{next\_to\_leading\_log\_expansion}(z) = \\ b_{1,1} - b_{1,1}\frac{\partial F_{\text{one\_term\_chord}}}{\partial z}(-b_{1,0}z) - \frac{b_{2,0}}{b_{1,0}}\frac{\partial F_{\text{one\_term\_of\_dec\_2}}}{\partial z}(-b_{1,0}z) \\ -\frac{b_{1,1}}{b_{1,0}}\frac{\partial F_{\text{two\_consec\_term}}}{\partial z}(-b_{1,0}z) - \frac{b_{2,0}}{b_{1,0}^{2}}\frac{\partial F_{\text{one\_dec\_2\_and\_one\_term}}}{\partial z}(-b_{1,0}z)\end{gathered}}$$

where $F_{\text{one\_term\_of\_dec\_2}}$, $F_{\text{two\_consec\_term}}$, $F_{\text{one\_dec\_2\_and\_one\_term}}$ are exponential generating functions of ω_{Δ}-marked decorated connected diagrams:

→ for $F_{\text{one\_term\_of\_dec\_2}}$: with only 1 terminal chord, this chord has decoration 2 *

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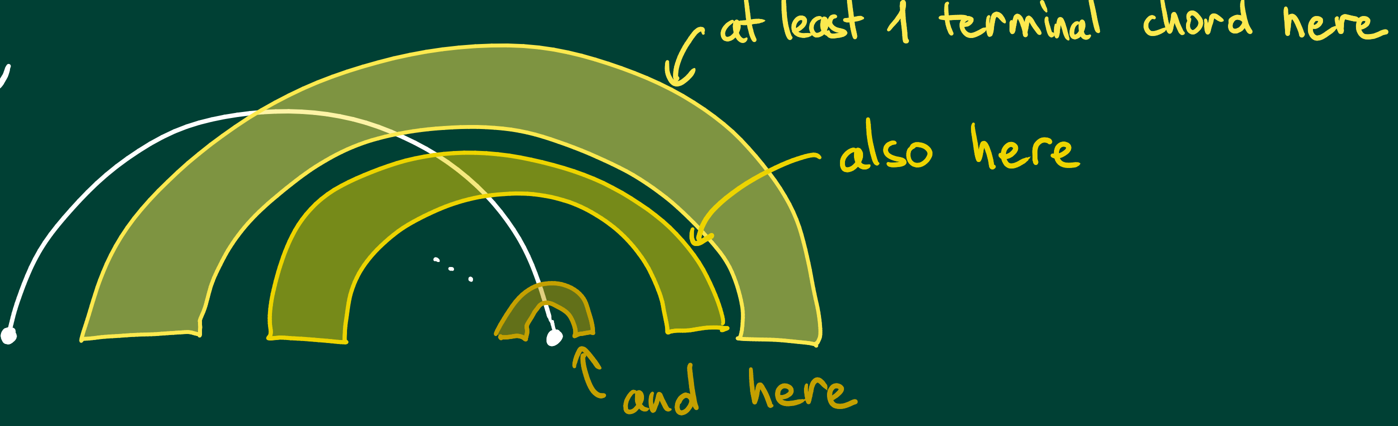
Similar formulas exist for the next-to^k leading log expansions

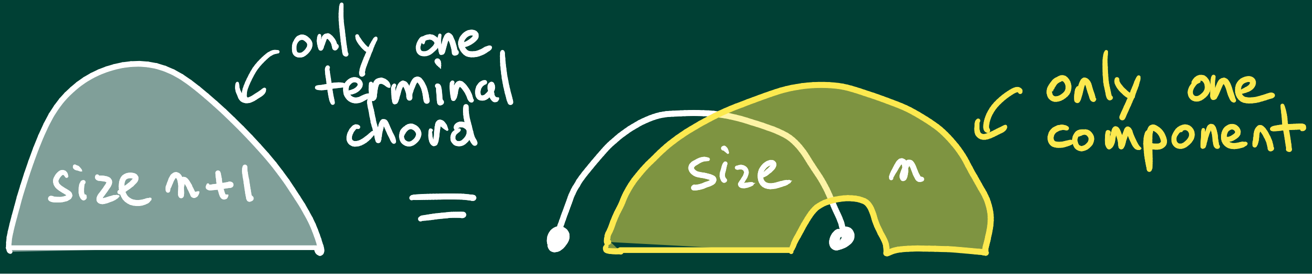
WITH ONE TERMINAL CHORD ($s = 2$)

WITH ONE TERMINAL CHORD ($\delta = 2$)

In general,



Recursive description of ω_2-marked connected chord diagrams with 1 terminal chord (there is no mark)



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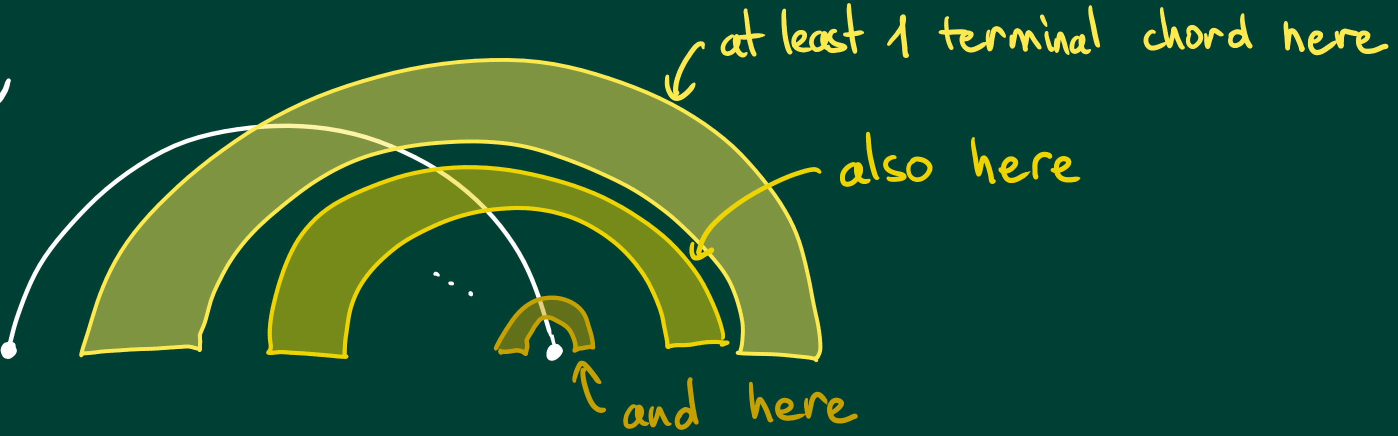

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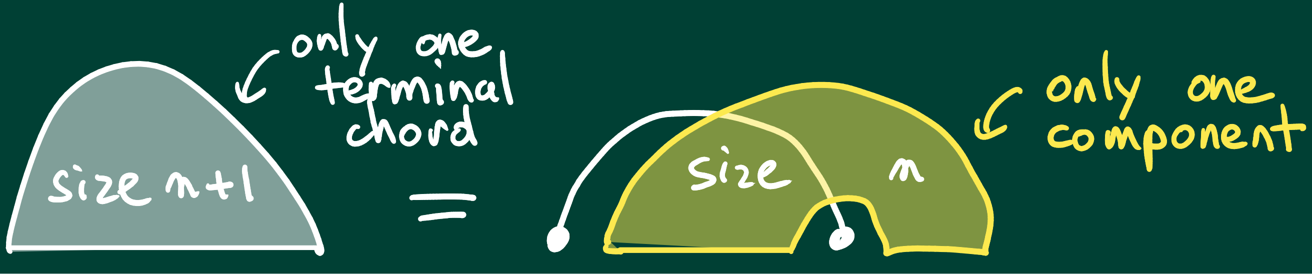

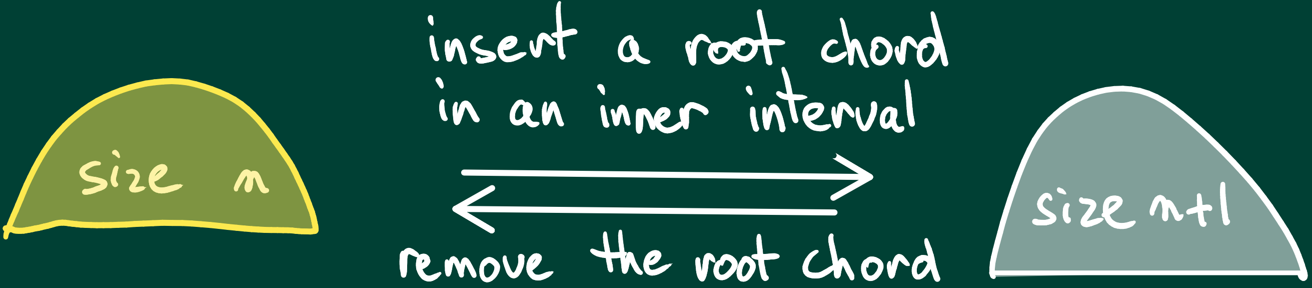

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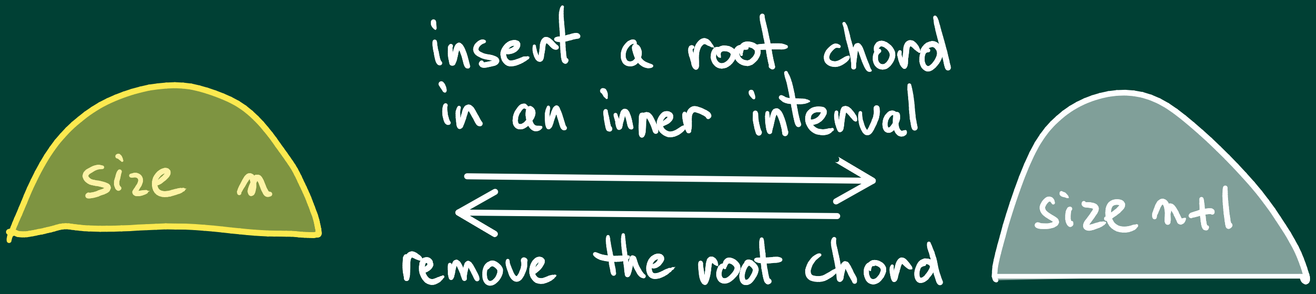

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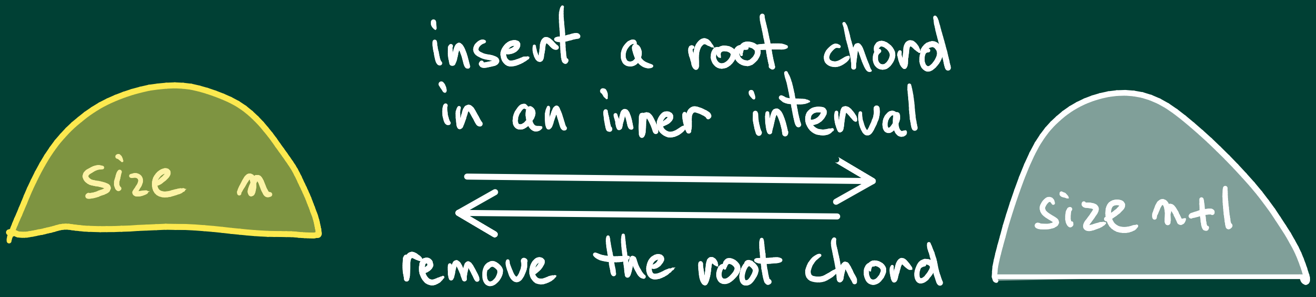
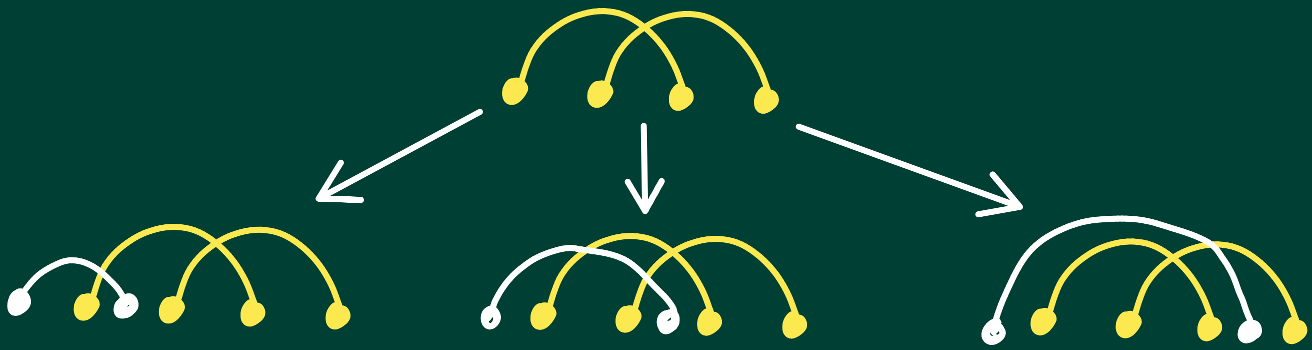


If a_n = number of ω_2-marked connected diagrams with n chords, only one of which is terminal,

$$a_{n+1} = (2n-1)\, a_n$$



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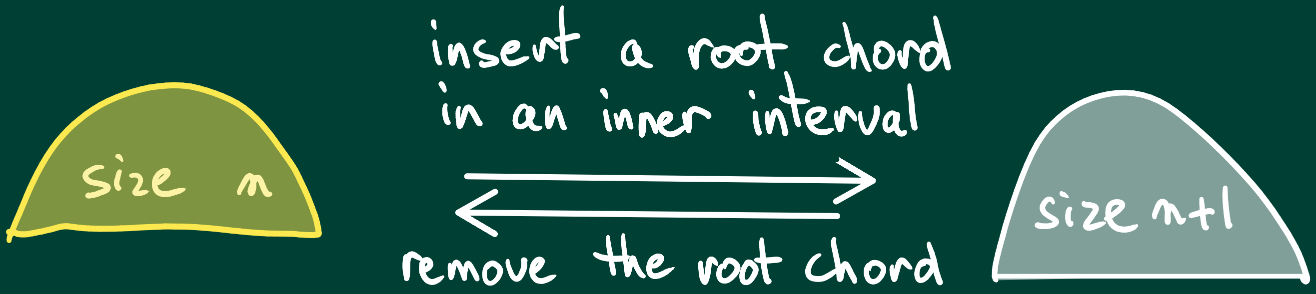

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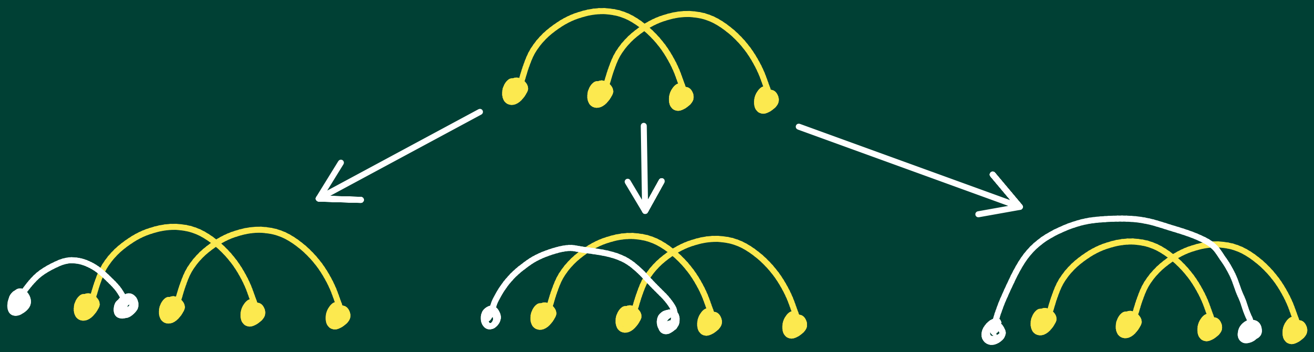

$$\boxed{a_n = (2n-3)!!}$$

WITH ONE TERMINAL CHORD ($\Delta = 3$)

What do we do when $\Delta = 3$?

The intervals covered by any chord must contain exactly 1 mark ...

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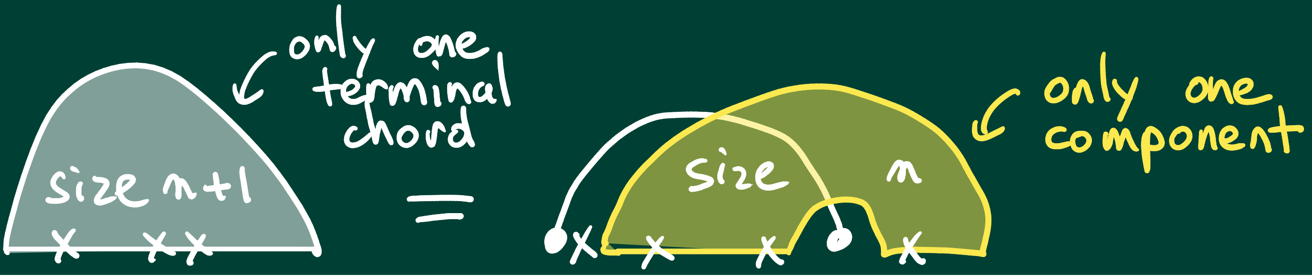

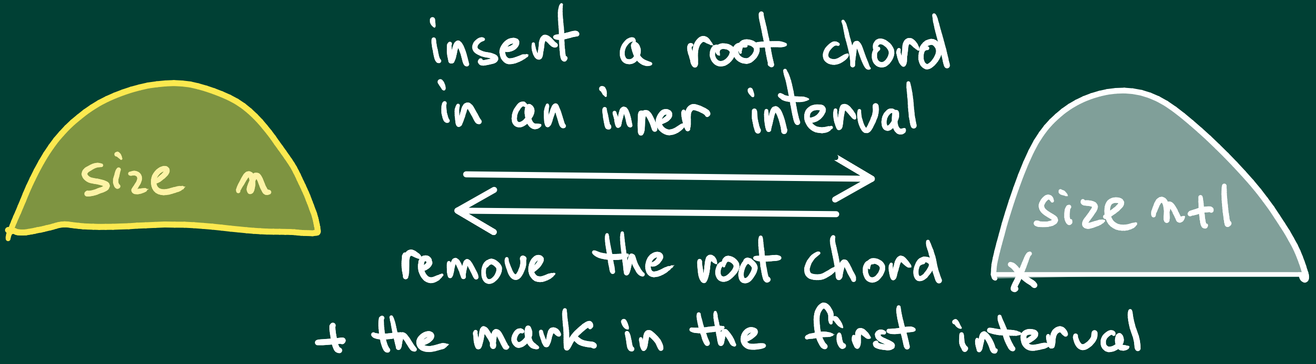

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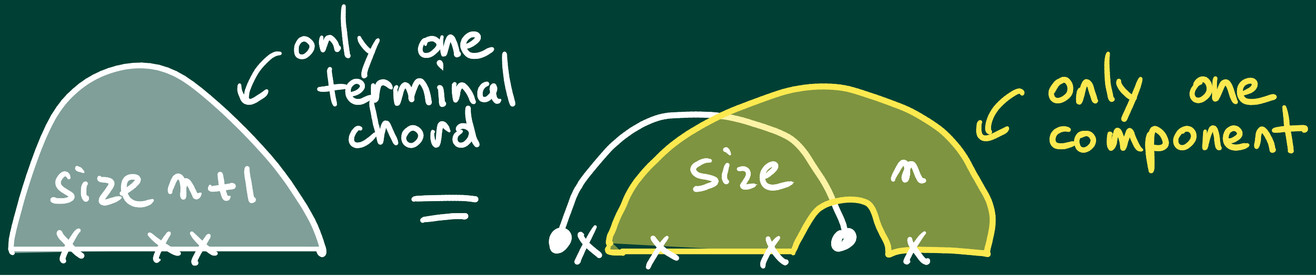

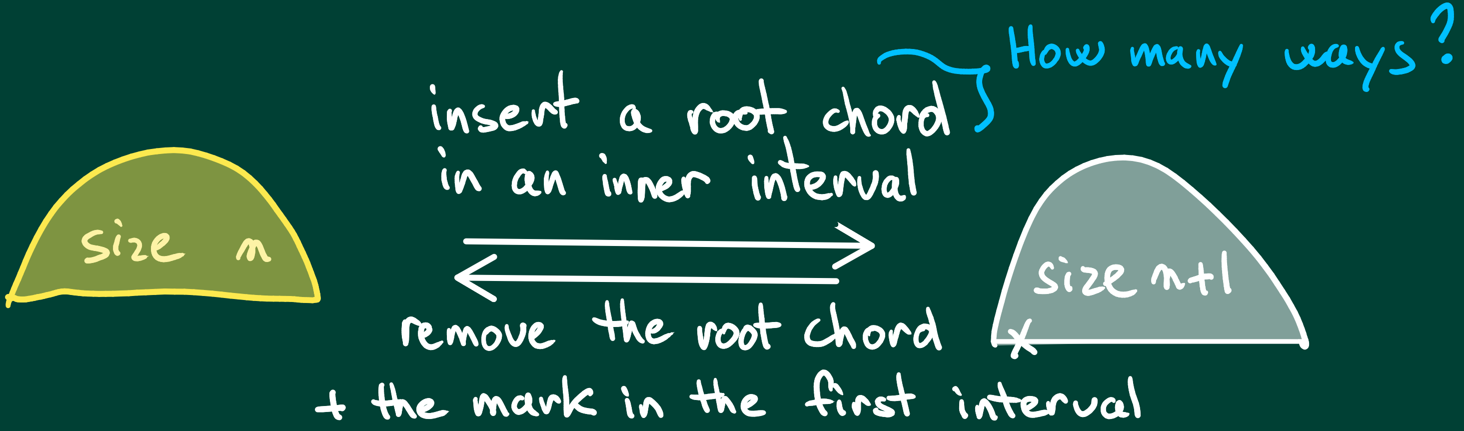

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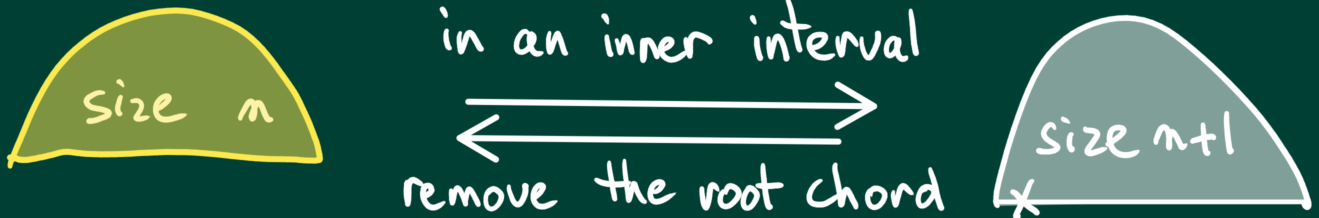
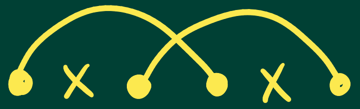


insert a root chord
in an inner interval

How many ways?

remove the root chord
+ the mark in the first interval



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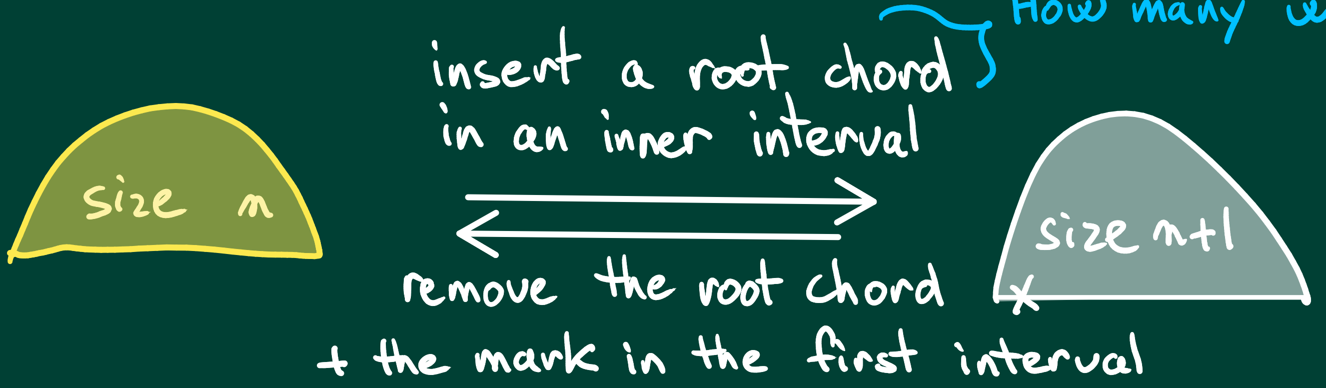

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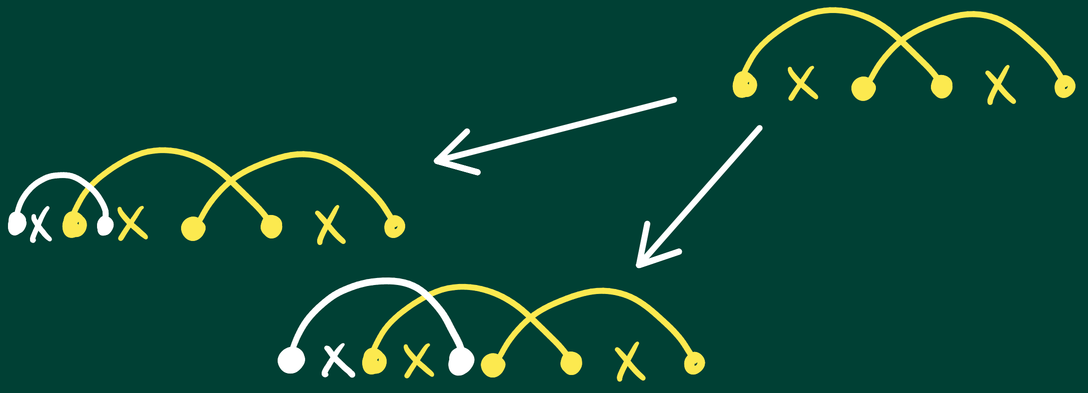

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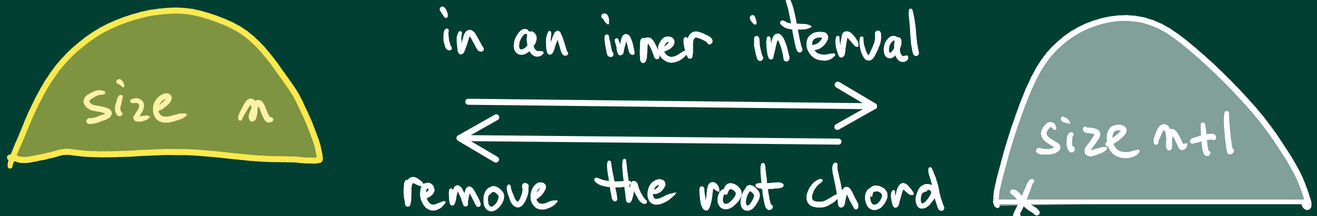

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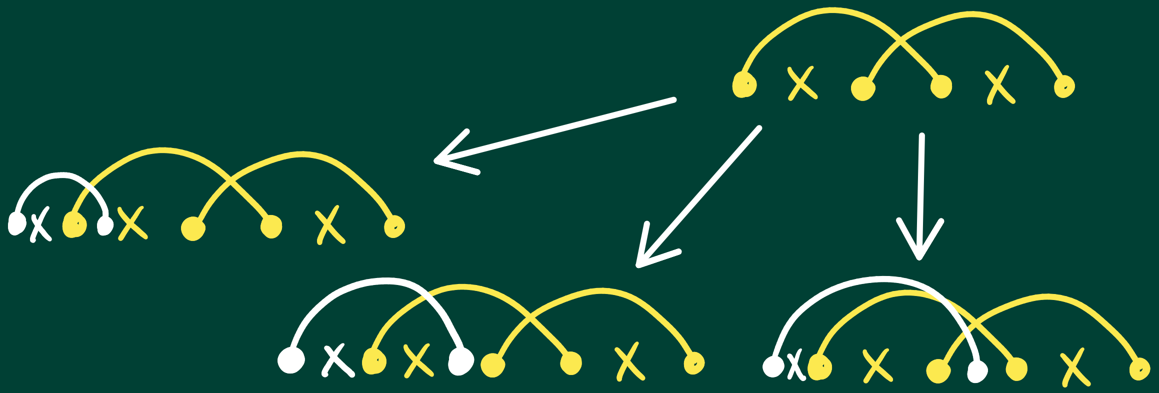

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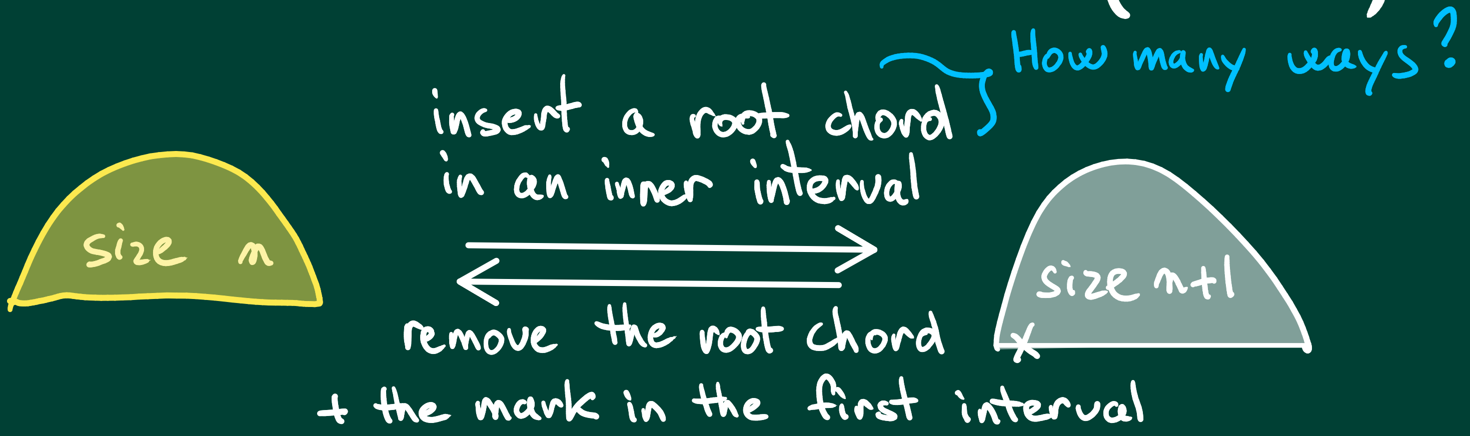

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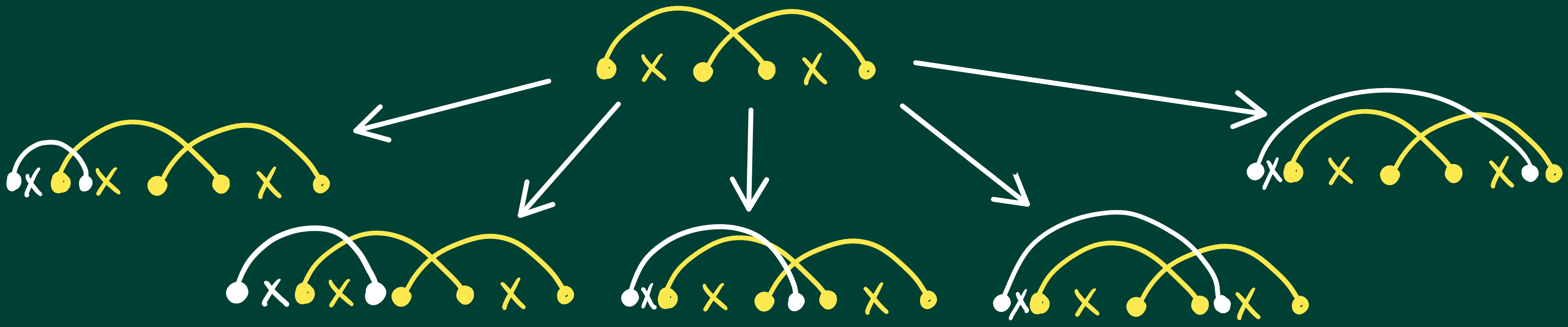

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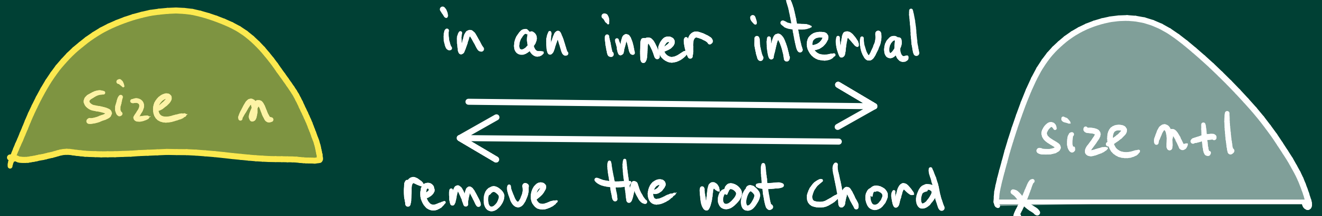

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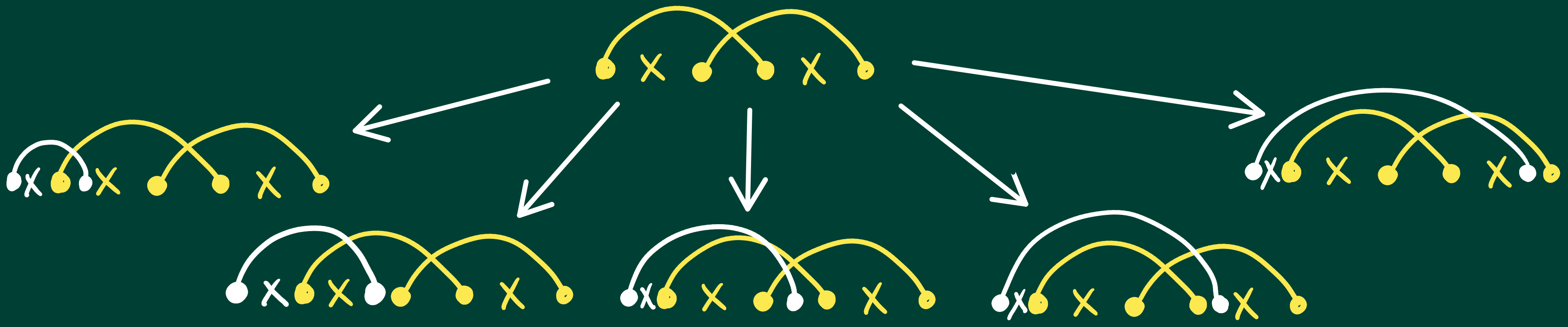

If b_n = number of ω_3-marked connected diagrams
with n chords, only one of which is terminal,

$$b_{n+1} = (3n-1)\, b_n$$

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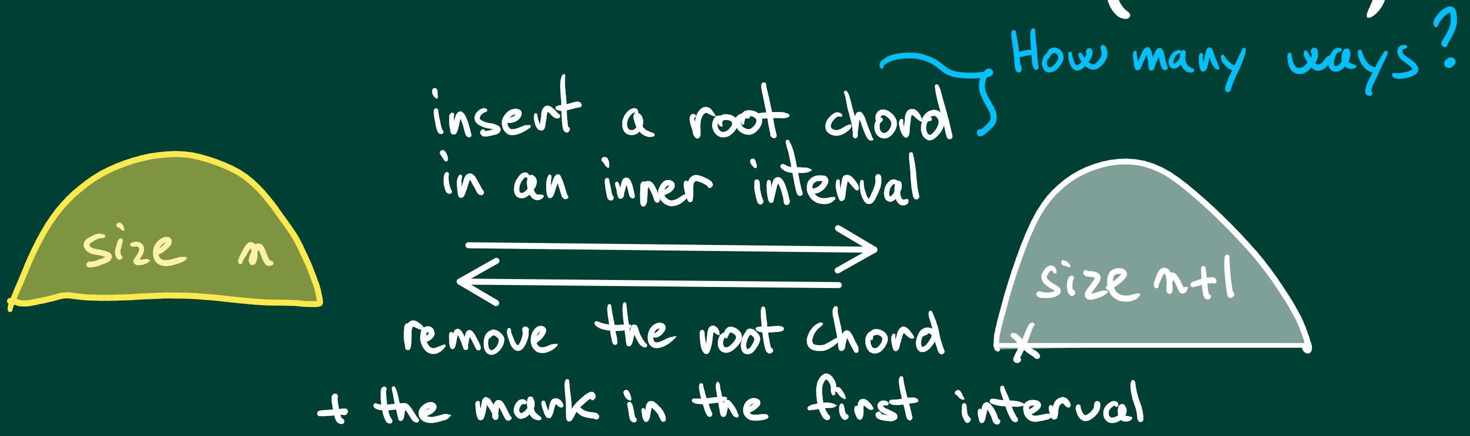

insert a root chord in an inner interval } How many ways?

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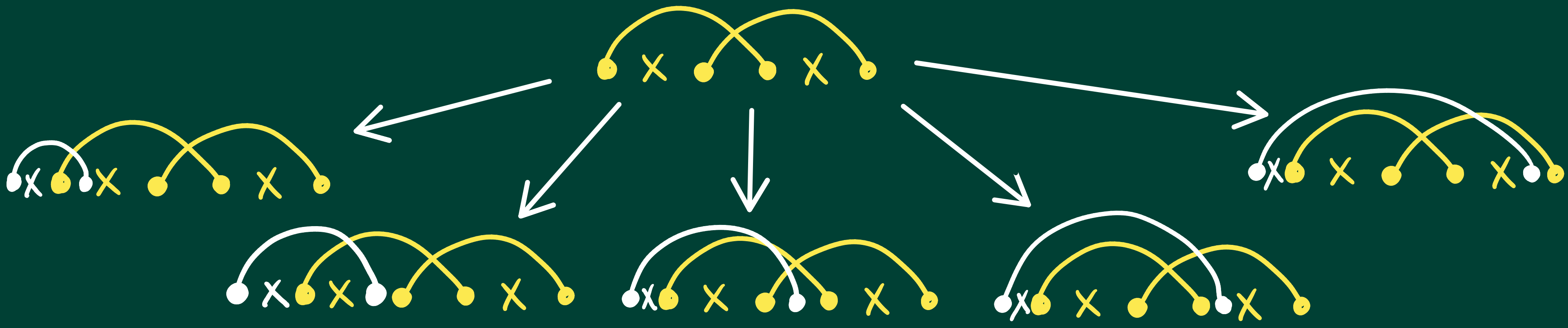

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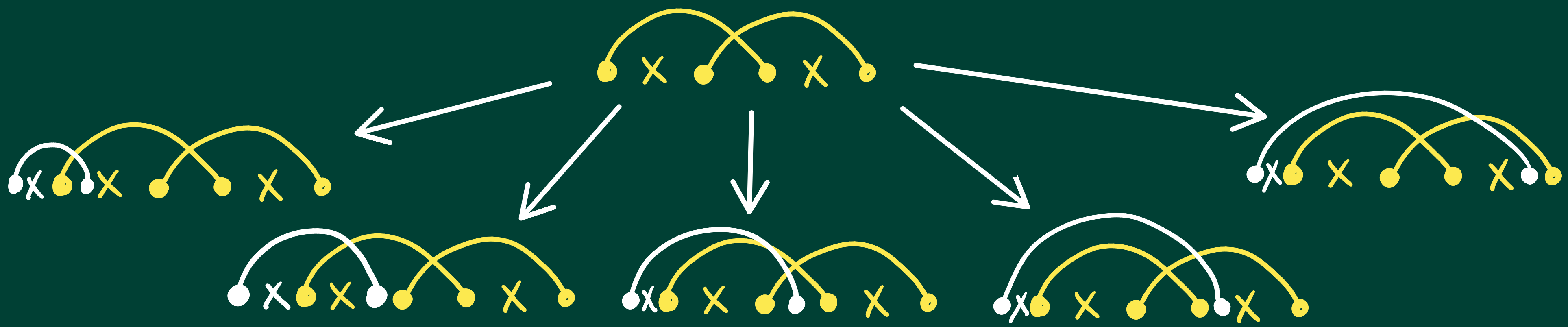
$$\boxed{b_n = (3n-4)!!! = (3n-4)\times(3n-7)\times\ldots\times 5\times 2}$$

MAGIC LEMMA



LEMMA

In general, for any $s \geq 1$, the number of ways to insert a root chord of decoration j in a ω_s-marked decorated connected diagram of size n is always

$$s \times n - 1$$

(= number of gaps between a dot/mark and a dot/mark)

MAGIC LEMMA

$s = 2$

size 11

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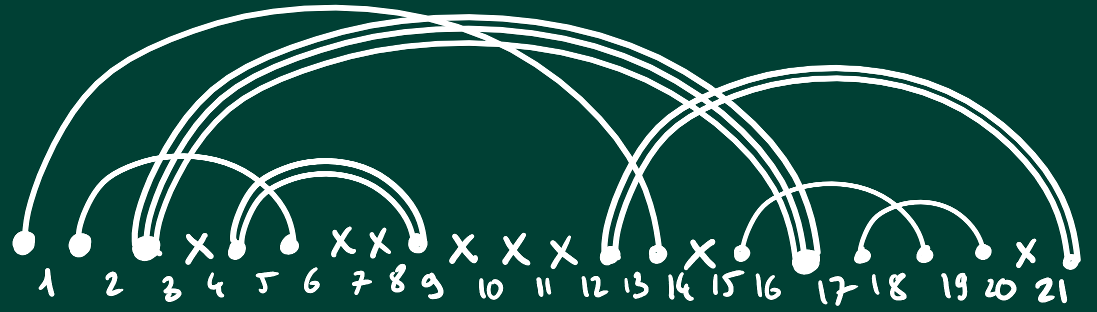

size 11 → 21 ways to insert a root chord

LEMMA

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LEADING LOG EXPANSIONS IN TERMS OF CHORD DIAGRAMS

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where $F_{\text{one\_term\_chord}}(z)$ = exponential generating function of ω_Δ-marked connected diagrams with only one terminal chord (all decorations = 1)

$$\boxed{\begin{aligned} &\text{next\_to\_leading\_log\_expansion}(z) = \\ &\beta_{1,1} - \beta_{1,1}\frac{\partial F_{\text{one\_term\_chord}}}{\partial z}(-\beta_{1,0}z) - \frac{\beta_{2,0}}{\beta_{1,0}}\frac{\partial F_{\text{one\_term\_of\_dec\_2}}}{\partial z}(-\beta_{1,0}z) \\ &- \frac{\beta_{1,1}}{\beta_{1,0}}\frac{\partial F_{\text{two\_consec\_term}}}{\partial z}(-\beta_{1,0}z) - \frac{\beta_{2,0}}{\beta_{1,0}^2}\frac{\partial F_{\text{one\_dec\_2\_and\_one\_term}}}{\partial z}(-\beta_{1,0}z) \end{aligned}}$$

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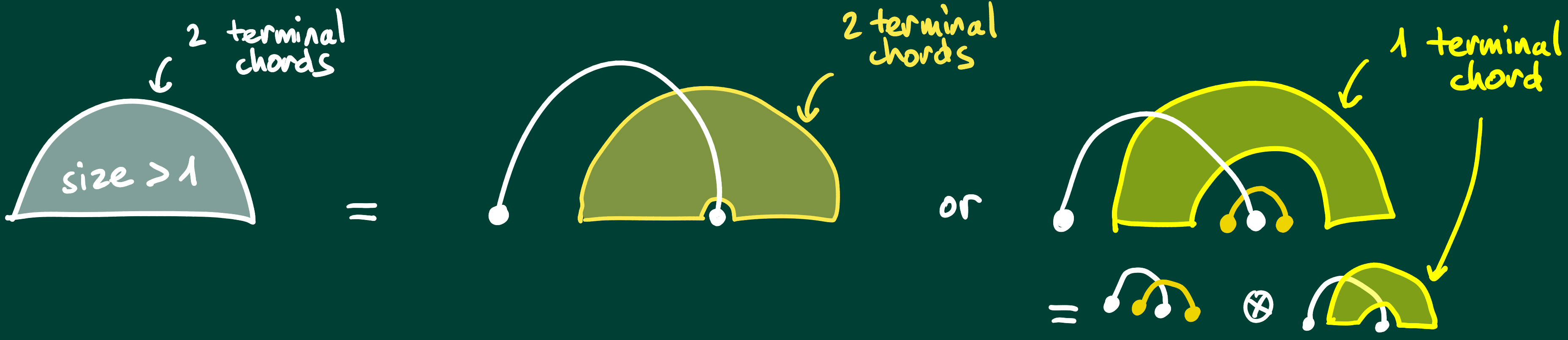
→ for F_one_dec_2_and_one_term : with only 1 terminal chord (decoration 1) and only one chord of decoration 2. *

* : the other chords have decoration 1

Similar formulas exist for the next-to^k leading log expansions

OTHER DECOMPOSITIONS

c_n = number of ω_s-marked connected diagrams with n chords with 2 terminal chords: last and before last for the intersection order



$$c_{n+1} = (sn-1)\,c_n + (s-1)\times(sn-s-1)\underbrace{!!\dots!}_{s}$$

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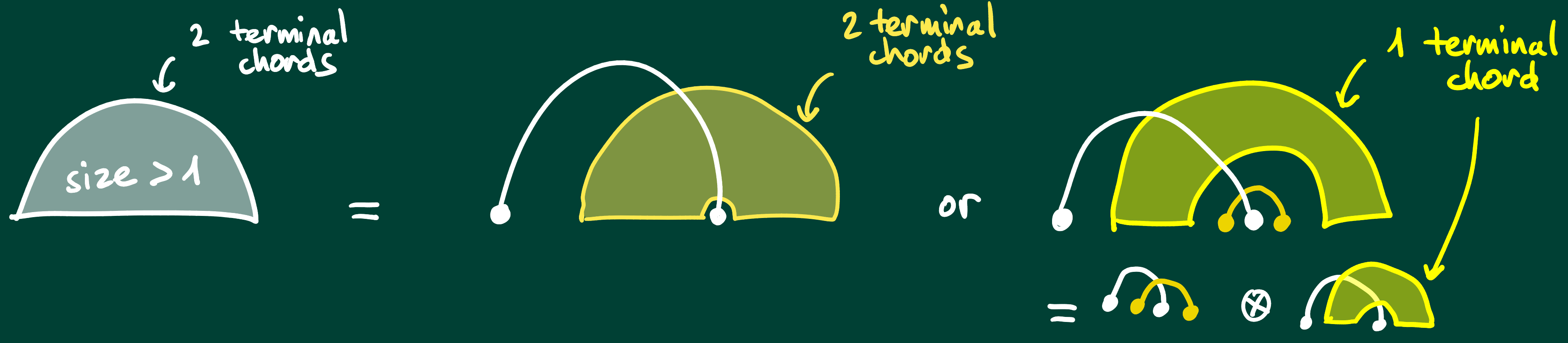
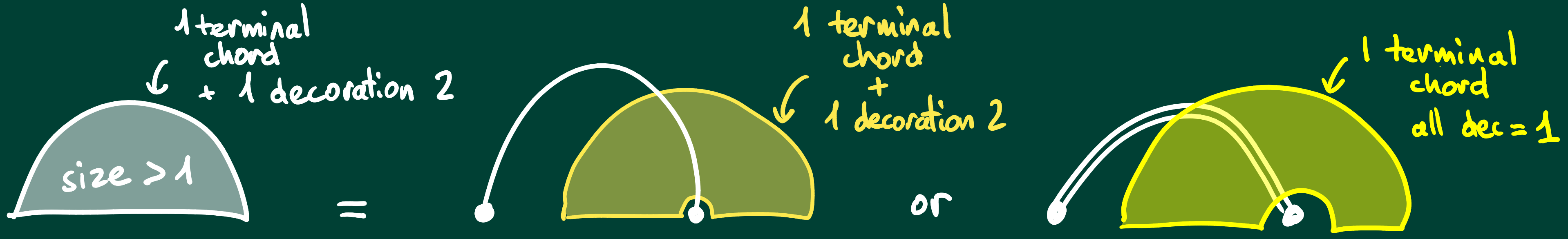


$$c_{n+1} = (s n - 1)\, c_n + (s-1) \times (s n - s - 1)\underbrace{!!\dots!}_{s}$$

---

d_n = number of ω_s-marked decorated connected diagrams with n chords with 1 terminal chord and where the only chord of decoration $\neq 1$ has decoration 2 and is not terminal.



$$d_{n+1} = (s n - 1)\, d_n + (s n - s - 1) \times (s n - 2s - 1)\underbrace{!!\dots!}_{s}$$

Part IV

Full results and conclusion

COMPUTING THE NEXT-TO^k LOG EXPANSIONS

All these decompositions can be used to compute the next-to^k leading log expansions.

Strategy: Decomposition → Recurrence → Differential equations for the exp. generating function → Solve them

$s=2$

Expansion	Expression
$H_0(z)$	$\sqrt{1+2\,f_{1,0}\,z}-1$
$H_1(z)$	$-\frac{-\left(f_{2,0}+f_{1,1}f_{1,0}\right)\,\ln\left(1+2\,zf_{1,0}\right)}{2f_{1,0}\,\sqrt{1+2\,za_{1,0}}}$
$H_2(z)$	$\frac{\left(f_{2,0}+f_{1,1}f_{1,0}\right)^2\left(\ln\left(1+2\,zf_{1,0}\right)\right)^2}{8\left(1+2\,zf_{1,0}\right)^{3/2}{f_{1,0}}^2}-\frac{\left(f_{2,0}+f_{1,1}f_{1,0}\right)^2\ln\left(1+2\,zf_{1,0}\right)}{2\left(1+2\,zf_{1,0}\right)^{3/2}{f_{1,0}}^2}-\frac{z\left(-{f_{2,0}}^2+3\,{f_{1,0}}^3f_{1,2}+3\,f_{2,1}{f_{1,0}}^2-f_{1,1}f_{2,0}f_{1,0}+f_{3,0}f_{1,0}\right)}{\left(1+2\,zf_{1,0}\right)^{3/2}f_{1,0}}$

Recovering
(and extending)
(and correcting)
[Krüger Kreimer 2015]

Every next-to^k leading log expansion can be automatically computed.
(We did it in maple.)

COMPUTING THE NEXT-TO^k LOG EXPANSIONS

All these decompositions can be used to compute the next-to^k leading log expansions.

Strategy: Decomposition → Recurrence → Differential equations for the exp. generating function → Solve them

Expansion	Expression
$H_0(z)$	$-f_{1,0}z$
$H_1(z)$	$-\frac{\ln(1+zf_{1,0})\,f_{2,0}}{f_{1,0}}$
$H_2(z)$	$-\frac{{f_{2,0}}^2\ln(1+zf_{1,0})}{{f_{1,0}}^2(1+zf_{1,0})}-\frac{\left(-{f_{2,0}}^2+f_{3,0}f_{1,0}+f_{2,1}{f_{1,0}}^2\right)z}{f_{1,0}(1+zf_{1,0})}$
$H_3(z)$	$-\frac{2\,{f_{1,0}}^4f_{2,2}+2\,{f_{1,0}}^3f_{3,1}-{f_{1,0}}^2f_{2,0}f_{2,1}+{f_{1,0}}^2f_{4,0}-2\,f_{1,0}f_{2,0}f_{3,0}+f_{2,0}^3}{2{f_{1,0}}^3}-\frac{f_{2,0}\left({f_{1,0}}^2f_{2,1}+f_{1,0}f_{3,0}-{f_{2,0}}^2\right)}{{f_{1,0}}^3(zf_{1,0}+1)}+\frac{{f_{2,0}}^3(\ln(zf_{1,0}+1))^2}{2(zf_{1,0}+1)^2{f_{1,0}}^3}+\frac{\left(-2\,f_{1,0}f_{2,0}f_{3,0}-2\,{f_{1,0}}^2f_{2,0}f_{2,1}\right)\ln(zf_{1,0}+1)}{2(za_{1,0}+1)^2{f_{1,0}}^3}+\frac{2\,{f_{1,0}}^4f_{2,2}+2\,{f_{1,0}}^3f_{3,1}+{f_{1,0}}^2f_{4,0}+{f_{1,0}}^2f_{2,0}f_{2,1}-{f_{2,0}}^3}{2(zf_{1,0}+1)^2{f_{1,0}}^3}$

$s=1$

Recovering (and extending) (and correcting) [Krüger Kreimer 2015]

Every next-to^k leading log expansion can be automatically computed.

(We did it in maple.)

THEOREM

ASYMPTOTIC DOMINANCE

$\underline{k}$ fixed and size $n \to +\infty$

The diagrams that have a positive asymptotic impact in the next-to^{k} leading log expansion have :

$\delta \geq 2$	$\delta = 1$
- k_1 terminal chords, consecutive and last for the intersection order - k_2 non-terminal chords of decoration 2 (decoration of the rest = 1) - $k_1 + k_2 = k$ In this case, number of such diagrams of size n $\sim \dfrac{(\delta-1)^{k_1}}{\Gamma(1-\delta)\, k!} \dbinom{k}{k_1} \ln(n)^{k} n^{-\frac{1}{\delta}-1} \delta^{n-k-1} n!$	- 1 terminal chord, of decoration 2 - k non-terminal chords of decoration 2 (decoration of the rest = 1) In this case, number of such diagrams of size n $\sim \dfrac{1}{(k-1)!} \ln(n)^{k-1} \times n^{-2} \times n!$

THEOREM

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Dichotomy between $s = 1$ and $s \geq 2$?!

ASYMPTOTIC ESTIMATE IN THE NEXT-TO^{k} LEADING-LOG EXPANSION

THEOREM

Suppose $\beta_{1,0} \neq 0$ and $\beta_{2,0} + (s-1)\,\beta_{11}\,\beta_{10} \neq 0$

Then the n^{th} coefficient in the next-to^{k} leading-log expansion (k fixed) is equivalent to:

$$\sim \frac{(-1)^{n}}{\Gamma(1-\frac{1}{s})\,k!}\,\beta_{1,0}^{\,n-k}\left(\beta_{2,0} + (s-1)\,\beta_{1,1}\,\beta_{1,0}\right)^{k} \ln(n)^{k}\; n^{k-\frac{1}{s}-1}\, s^{n-1}$$

$(s \geq 2)$

$$\sim \frac{(-1)^{n}}{(k-1)!}\,\beta_{1,0}^{\,n-k}\,\beta_{2,0}^{\,k}\,\ln(n)^{k-1}\; n^{k-2} \qquad (s=1)$$

Only $\beta_{1,0}$, $\beta_{1,1}$ and $\beta_{2,0}$ asymptotically matter!

OPEN QUESTIONS

→ Direct interpretation of the solution of $G(x, L)$ in terms of Feynman graphs?

→ Are there better combinatorial objects than ω_Δ-marked decorated connected chord diagrams?

→ Studying $[x^n L^m] G(x, L)$ in other directions than $n = m + k$ (k fixed)

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