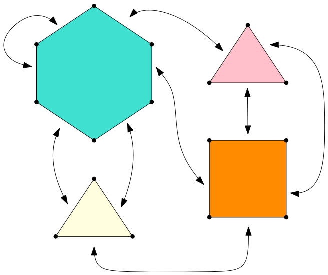
b-weighted constellations of Chapuy-Dołęga

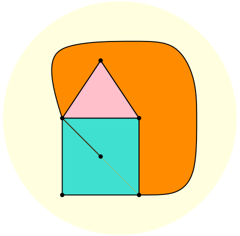
Valentin Bonzom
with **V. Nador** (LaBRI U. Bordeaux-LIPN U. Sorbonne Paris Nord)

LIGM – Université Gustave Eiffel

Séminaire Philippe Flajolet
28 septembre 2023

Take polygons and glue into surface

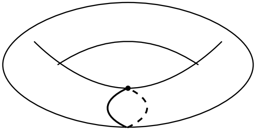




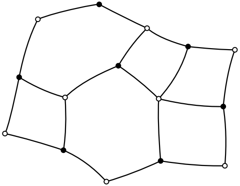
- More structure than graphs
- Vertices, edges and **faces** = polygons
- They are topological surfaces
- Nice interplay between combinatorics and topology
- Euler's formula for the genus $g \geq 0$

$$F - E + V = 2 - 2g$$

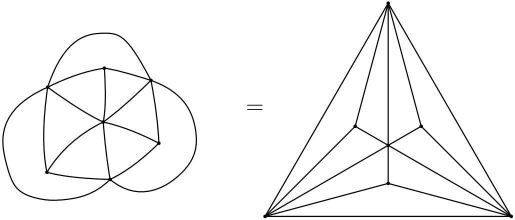
Examples of maps



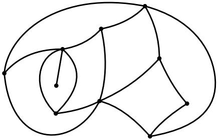
Not a map



Planar bipartite map

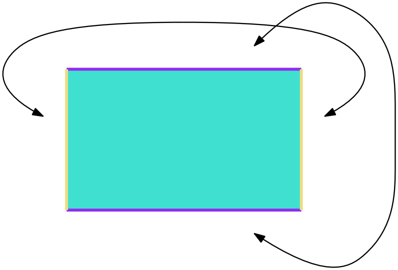


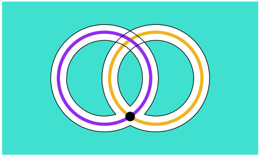
Planar triangulation



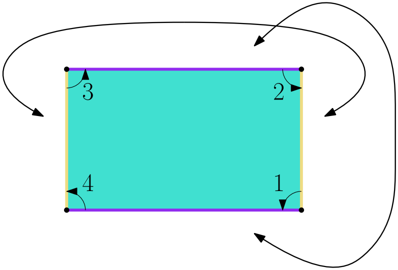
Planar quadrangulation

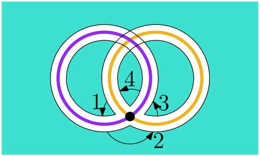
Non-zero genus



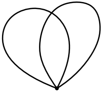


Non-zero genus

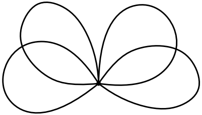




Drawing in the plane of map with genus: Crossings



Map of genus 1



Map of genus 2

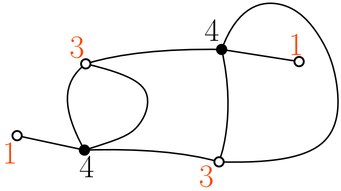
Short history

- Paléocartique : Tutte [60s] pioneered their enumeration and found remarkable formula for planar maps with n edges

$$M_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

  Suggest something deep about their structure/encoding
- Mésocartique : Connections to quantum field theory and integrable systems by physicists, topological recursion, maps of non-zero genus
- Néocartique : Schaeffer's bijections and many more, use of distances and applications to convergence to random metric spaces
- Map holocene?
- Today : Phenomena which relate maps of different genus ($\sim$ mésocartique)

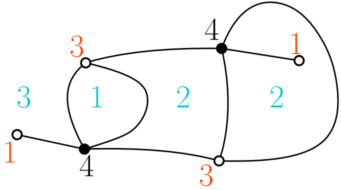
- A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$ like $(3, 2, 2, 1)$
- Encode the degrees of white vertices in a partition



$$\begin{cases} \lambda_\circ = (3, 3, 1, 1) \\ \lambda_\bullet = (4, 4) \\ \end{cases}$$

- In a bipartite map, the numbers of sides of faces are even
- Face degree: half boundary length

- A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$ like $(3, 2, 2, 1)$
- Encode the degrees of white vertices in a partition



$$\begin{cases} \lambda_\circ = (3, 3, 1, 1) \\ \lambda_\bullet = (4, 4) \\ \lambda_{\text{faces}} = (3, 2, 2, 1) \end{cases}$$

- In a bipartite map, the numbers of sides of faces are even
- Face degree: half boundary length

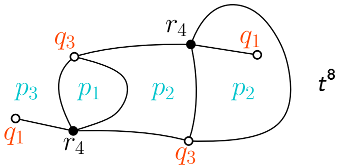
Defining the generating functions

- Let $\vec{p} = (p_1, p_2, \dots)$ an infinite set of indeterminates

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l} \qquad\qquad p_{(3,2,2,1)} = p_3 p_2^2 p_1$$

- Denote $B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}})$ the number of bipartite maps with n labeled edges, white vertex degrees by $\lambda_\circ$ and black vertex degrees by $\lambda_\bullet$ and face degrees by λ_{faces}

$$B^{(3)}(t, \vec{p}, \vec{q}, \vec{r}) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\lambda_{\text{faces}}, \lambda_\circ, \lambda_\bullet} B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}}) p_{\lambda_{\text{faces}}} q_{\lambda_\circ} r_{\lambda_\bullet}$$



- Too difficult to write an equation!

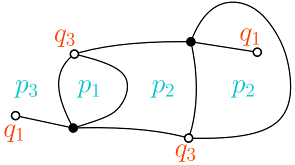
Defining the generating functions

- Let $\vec{p} = (p_1, p_2, \ldots)$ an infinite set of indeterminates

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l} \qquad\qquad p_{(3,2,2,1)} = p_3 p_2^2 p_1$$

- Denote $B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}})$ the number of bipartite maps with n labeled edges, white vertex degrees by $\lambda_\circ$ and black vertex degrees by $\lambda_\bullet$ and face degrees by λ_{faces}

$$B^{(2)}(t, u, \vec{p}, \vec{q}) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\lambda_{\text{faces}}, \lambda_\circ, \lambda_\bullet} B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}}) p_{\lambda_{\text{faces}}} q_{\lambda_\circ} u^{\ell(\lambda_\bullet)}$$



$t^8 u^2$

- Tough but not impossible! [Eynard, Bousquet-Mélou–Schaeffer, Chapuy-Dołęga, Albenque-Bouttier]

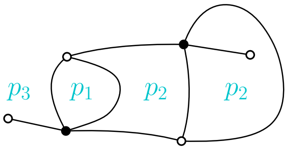
Defining the generating functions

- Let $\vec{p} = (p_1, p_2, \dots)$ an infinite set of indeterminates

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l} \qquad\qquad p_{(3,2,2,1)} = p_3 p_2^2 p_1$$

- Denote $B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}})$ the number of bipartite maps with n labeled edges, white vertex degrees by $\lambda_\circ$ and black vertex degrees by $\lambda_\bullet$ and face degrees by λ_{faces}

$$B(t,u,v,\vec{p}) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\lambda_{\text{faces}},\lambda_\circ,\lambda_\bullet} B_n(\lambda_\circ, \lambda_\bullet, \lambda_{\text{faces}}) p_{\lambda_{\text{faces}}} v^{\ell(\lambda_\circ)} u^{\ell(\lambda_\bullet)}$$

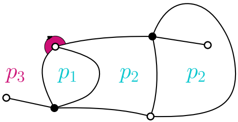


$t^8 u^2 v^4$

- Connected maps $H = \ln B(t,u,v,\vec{p})$

Rooted maps

- Root: marked oriented corner at white vertex
- Root edge: the one after the corner



- Let $H_i(t, u, v, \vec{p})$ be the GF of connected, rooted bip maps with a root face of degree i
- Rooting is choosing a corner in a face of degree i uniformly

$$H_i(t, u, v, \vec{p}) = i \frac{\partial}{\partial p_i} H(t, u, v, \vec{p})$$

b-deformed maps [Chapuy-Dołęga22]

- To every non-oriented, rooted map, associate a weight $\rho_b(M,c)$ def to follow!
- Weight of a non-rooted map $\boxed{\rho_b(M) = \frac{1}{n}\sum_{\text{roots}} \rho_b(M,c)}$

$$H^b(t,u,v,\vec{p}) = \sum_{\text{labeled } M} \frac{t^n}{2^{n-1}n!} p_{\lambda_{\text{faces}}} u^{\ell_\bullet} v^{\ell_\circ} \rho_b(M)$$

Theorem [VB-Nador, to appear]

- Hidden symmetry under rooting

$$H^b_i(t,u,v,\vec{p}) = i\frac{\partial}{\partial p_i} H^b(t,u,v,\vec{p})$$

- Duality

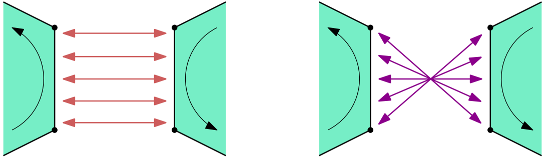
$$H^b_i(t,u,v,\vec{p}) = \tilde{H}^b_i(t,u,v,\vec{p})$$

calculated with b-weights of dual maps

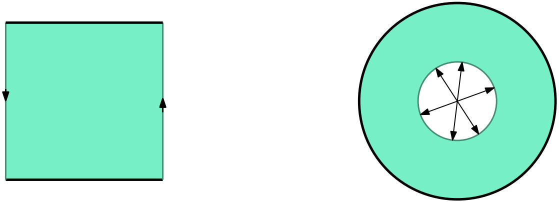
- Those results hold for maps, bip maps, 3-constellations and bip maps with controlled degree of white vertices ≤ 3

Non-oriented surfaces

- Two ways of gluing polygon sides

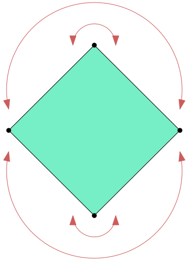


- Create crosscaps: disk replaced with Möbius strip

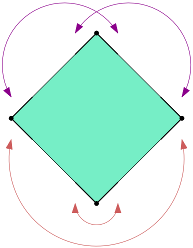


- Genus counts handles + crosscaps/2
- Genus 1: torus or 2 crosscaps (Klein bottle)
- Integer genus: n handles or $2n$ cross-caps

Examples

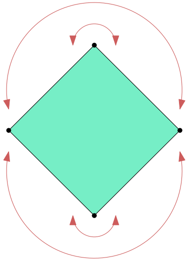


$\rightarrow ?$

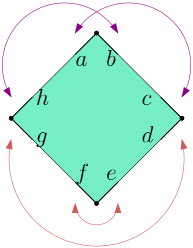


$\rightarrow ?$

Examples

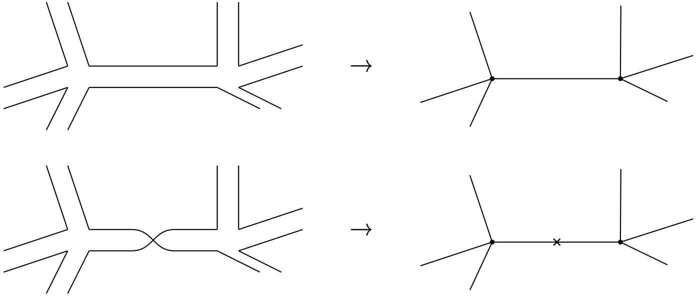


→?



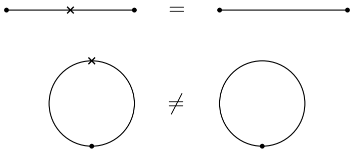
→?

Types of edges

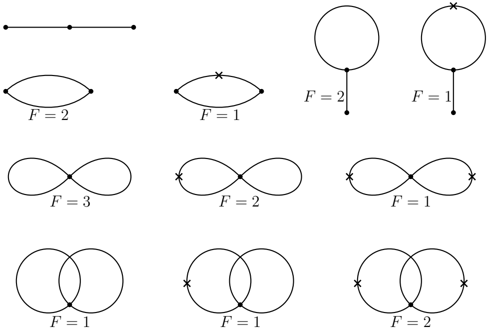


Beware

- The twisted edge is deceiptive



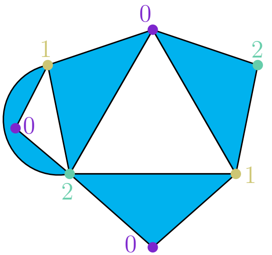
- Those with 2 edges



- Twists may increase, decrease or not change the number of faces

Non-oriented constellations [Chapuy-Dołęga22]

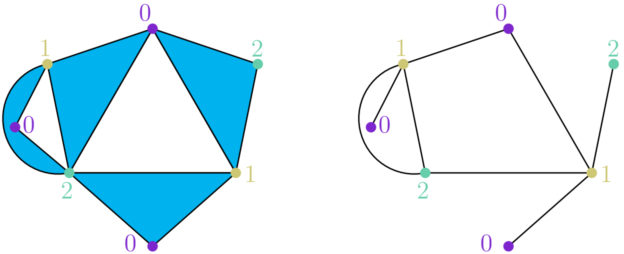
- $k \geq 1$
- Recall oriented constellations: hyperedges with colors $0, \ldots, k$ counter-clockwise



- Remove edges of colors $0k$ and extend faces
- Non-oriented constellations: allow for twisted edges

Non-oriented constellations [Chapuy-Dołęga22]

- $k \geq 1$
- Recall oriented constellations: hyperedges with colors $0, \ldots, k$ counter-clockwise



- Remove edges of colors $0k$ and extend faces
- Non-oriented constellations: allow for twisted edges

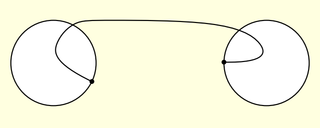
What happens when you add an edge

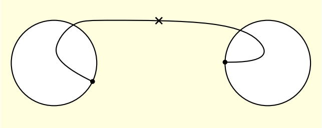
- b: formal variable

Define the b-weight of an edge [Chapuy-Dołęga22]

For an edge in a map

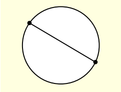
1. Connect two disjoint components: $\rho_b(M, e) = 1$
2. Merge two faces in the same c.c.





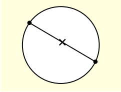
$\rightarrow 1 + b$

3. Split a face into two



$\rightarrow 1$

4. Add an twisted edge to a face, no change to face number

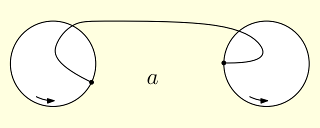


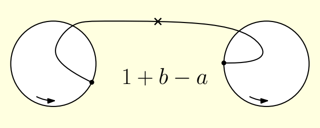
$\rightarrow b$

More on condition 2

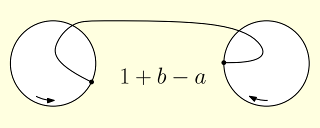
A simple way to realize it:

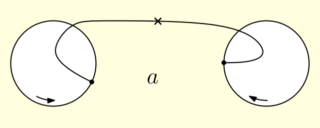
- ▷ Equip every connected map with a fixed orientation of its faces
- ▷ Use those orientations to distinguish the two types of edges





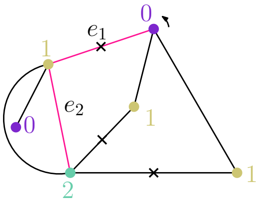
- ▷ and





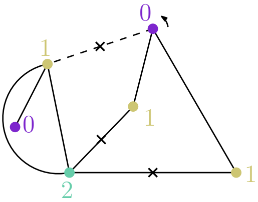
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$\rightarrow \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M \backslash \{e_1\}, e_2) \cdots$

- For a rooted M, collect the weights around the root vertex

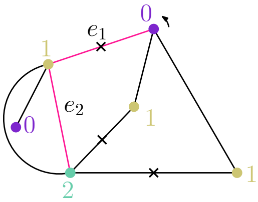


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n} \sum_c \rho_b(M, c)$$

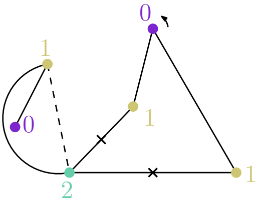
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$\to \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M\backslash\{e_1\}, e_2)\cdots$

- For a rooted M, collect the weights around the root vertex

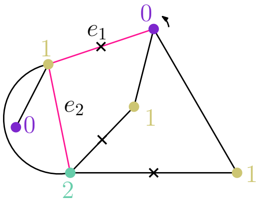


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n}\sum_{c} \rho_b(M, c)$$

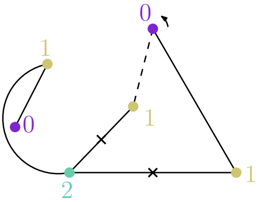
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$\rightarrow \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M \backslash \{e_1\}, e_2) \cdots$

- For a rooted M, collect the weights around the root vertex

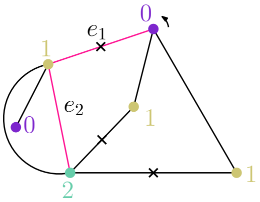


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n} \sum_c \rho_b(M, c)$$

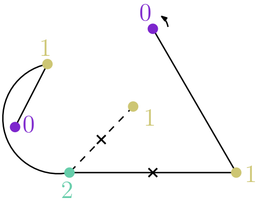
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$$\to \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M \backslash \{e_1\}, e_2) \cdots$$

- For a rooted M, collect the weights around the root vertex

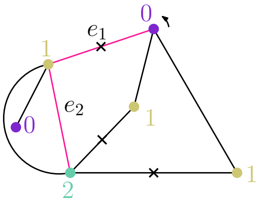


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n} \sum_c \rho_b(M, c)$$

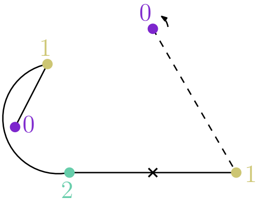
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$$\to \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M\backslash\{e_1\}, e_2)\cdots$$

- For a rooted M, collect the weights around the root vertex

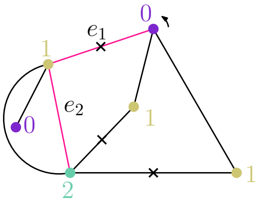


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n}\sum_c \rho_b(M, c)$$

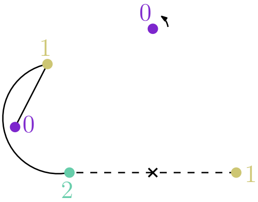
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$\to \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M \backslash \{e_1\}, e_2) \cdots$

- For a rooted M, collect the weights around the root vertex

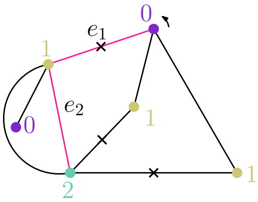


- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n} \sum_c \rho_b(M, c)$$

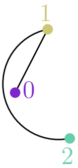
b-weight of a constellation [Chapuy-Dołęga22]

- k-path from the root: follow k edges along root face



$\to \rho_b(M; e_1, \ldots, e_k) = \rho_b(M, e_1)\rho_b(M \backslash \{e_1\}, e_2) \cdots$

- For a rooted M, collect the weights around the root vertex



- Repeat with the dual! Get $\rho_b(M, c)$. Then for labeled constellations

$$\rho_b(M) = \frac{1}{n} \sum_c \rho_b(M, c)$$

Questions

- Series from Chapuy-Dołęga

$$F^b(t, \vec{p}, \vec{q}, u_1, \dots, u_k) = \sum_{\text{labeled } M} \frac{t^n}{2^{n-1} n!} \rho_b(M) \prod_f p_{d_f} \prod_{\substack{\text{vertices} \\ \text{col } 0}} q_{d_v} \prod_{c=1}^{k} u_c^{V_c}$$

- They proved that for $H^b = (1+b) \ln F^b$

$$i \frac{\partial}{\partial q_i} H^b = H^b_{[i]}$$

with $H^b_{[i]}$ the GF of rooted constellations with root vertex of degree i

- Conjectured that

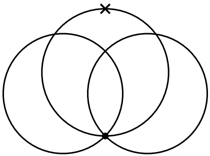
$$i \frac{\partial}{\partial p_i} H^b = H^b_i$$

- Isn't it obvious by duality?
- Duality exchanges $\vec{p}$ and $\vec{q}$, so

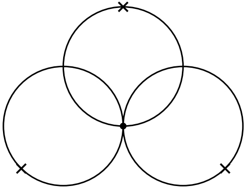
$$i \frac{\partial}{\partial p_i} H^b = \tilde{H}^b_i$$

Dual maps

- A non-orientable map and its dual



$$b^3 + ba + b(1 + b - a)$$



$$b^3 + 2b(1 + b - a)$$

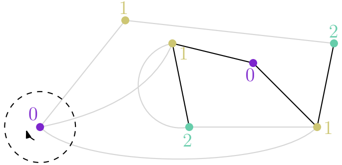
- $F = 1, V = 1, g = 3/2$

How to proceed?

- Prove $i\frac{\partial H^b}{\partial p_i} = H_i^b$
- $i\frac{\partial H^b}{\partial p_i}$ is a bit mysterious
- But H_i^b satisfies a combinatorial equation [Chapuy-Dołęga]

$$H_i^b = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + (1+b) \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} \right.\right.$$
$$\left.\left. + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + b \sum_l l y_l \frac{\partial}{\partial y_l} + \sum_{l,n} y_{n+l-1} \tilde{H}_l^b \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- Idea: delete the root vertex and its incident k-paths

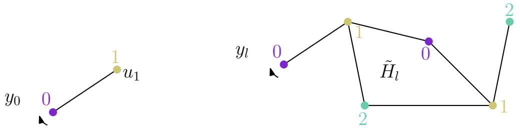


How to proceed?

- Prove $i\frac{\partial H^b}{\partial p_i} = H_i^b$
- $i\frac{\partial H^b}{\partial p_i}$ is a bit mysterious
- But H_i^b satisfies a combinatorial equation [Chapuy-Dołęga]

$$H_i^b = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + (1+b) \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} \right.\right.$$
$$\left.\left. + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + b \sum_l l y_l \frac{\partial}{\partial y_l} + \sum_{l,n} y_{n+l-1} \tilde{H}_l^b \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- First step is $m = 1, c = 1$

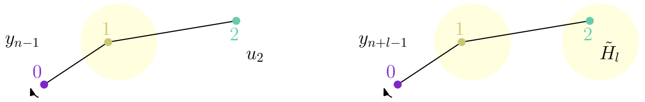


How to proceed?

- Prove $i\frac{\partial H^b}{\partial p_i} = H_i^b$
- $i\frac{\partial H^b}{\partial p_i}$ is a bit mysterious
- But H_i^b satisfies a combinatorial equation [Chapuy-Dołęga]

$$H_i^b = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + (1+b) \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} \right.\right.$$
$$\left.\left. + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + b \sum_l l y_l \frac{\partial}{\partial y_l} + \sum_{l,n} y_{n+l-1} \tilde{H}_l^b \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- From color c to $c+1$, 6 possibilities

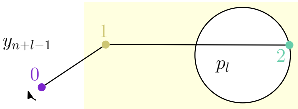


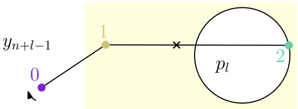
How to proceed?

- Prove $i\frac{\partial H^b}{\partial p_i} = H_i^b$
- $i\frac{\partial H^b}{\partial p_i}$ is a bit mysterious
- But H_i^b satisfies a combinatorial equation [Chapuy-Dołęga]

$$H_i^b = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + (1+b) \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + b \sum_l l y_l \frac{\partial}{\partial y_l} + \sum_{l,n} y_{n+l-1} \tilde{H}_l^b \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- From color c to $c+1$, 6 possibilities



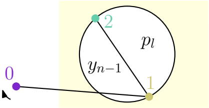


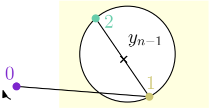
How to proceed?

- Prove $i\frac{\partial H^b}{\partial p_i} = H_i^b$
- $i\frac{\partial H^b}{\partial p_i}$ is a bit mysterious
- But H_i^b satisfies a combinatorial equation [Chapuy-Dołęga]

$$H_i^b = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + (1+b) \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} \right.\right.$$
$$\left.\left. + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + b \sum_l l y_l \frac{\partial}{\partial y_l} + \sum_{l,n} y_{n+l-1} \tilde{H}_l^b \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- From color c to $c+1$, 6 possibilities





Evolution equation

- Show that RHS is $i\frac{\partial H^b}{\partial p_i}$? No combinatorial proof. Instead, a lot of algebra!
- From [Chapuy-Dołęga], evolution equation

$$\frac{\partial F}{\partial t} = \sum_m q_m t^{m-1} K_m F$$

In the orientable case

- From double counting $t\frac{\partial H}{\partial t} = \sum_{i\geq 1} ip_i \frac{\partial H}{\partial p_i}$
- We know that $i\frac{\partial H}{\partial p_i} = H_i$. Implies $t\frac{\partial H}{\partial t} = \sum_i p_i H_i$

$$H_i = \sum_m q_m t^m [y_{i-m}] \left(\prod_{c=1}^{k} \left(u_c + \sum_{l,n} y_{l+n-1} \frac{l\partial^2}{\partial p_l \partial y_{n-1}} + \sum_{l,n} y_{n-1} p_l \frac{\partial}{\partial y_{n+l-1}} + \sum_{l,n} y_{n+l-1} H_l \frac{\partial}{\partial y_{n-1}} \right) \right)^m y_0$$

- It comes for $H = \ln F$

$$\frac{\partial F}{\partial t} = \sum_{i,m\geq 1} p_i q_m t^{m-1} M_{i,m} F = \sum_{m\geq 1} q_m t^{m-1} K_m F$$

From evolution equation to constraints

- For $b = 0$ we went from $i\frac{\partial H^b}{\partial p_i} = H_i^b$ to evolution equation
- Evolution equation determines the solution, so can we go back?

Lemma [VB-Chapuy-Dołęga] extension [VB-Nador]

- Some assumptions easy to check
- Evolution equation $t\frac{\partial F}{\partial t} = \sum_{i\geq 1} p_i M_i F$

$$\sum_{i\geq 1} p_i \underbrace{\left(i\frac{\partial}{\partial p_i} - M_i\right)}_{L_i} F = 0$$

- "Closed algebra with left structure operators"

$$[L_i, L_j] = \sum_{k\geq 1} D_{ijk} L_k$$

- Then for all $i \geq 1$

$$L_i F = 0$$

Examples of use

- Denote $p_i^* = i\frac{\partial}{\partial p_i}$
- Bipartite maps

$$L_i^{\text{bip}} = -p_i^* + (1+b)t \sum_{\substack{l,m\geq 1\\ l+m=i-1}} p_l^* p_m^* + t\sum_{l\geq 1} p_l p_{l+i-1}^* + t\Big(b(i-1)+u_1+u_2\Big)p_{i-1}^* + t\frac{u_1 u_2}{1+b}\delta_{i,1}$$

- General maps

$$L_i^{\text{maps}} = -p_i^* + (1+b)t^2 \sum_{\substack{n,m\geq 1\\ n+m=i-2}} p_n^* p_m^* + t^2\sum_{n\geq 1} p_n p_{n+i-2}^* + t^2(b(i-1)+2u)p_{i-2}^* + \frac{t^2 u}{1+b}p_1\delta_{i,1} + \frac{t^2 u(b+u)}{1+b}\delta_{i,2}$$

- In both cases [Adler-van Moerbeke], half-Virasoro

$$[L_i^{\text{bip}}, L_j^{\text{bip}}] = t(i-j)L_{i+j-1}^{\text{bip}}, \qquad [L_i^{\text{maps}}, L_j^{\text{maps}}] = t^2(i-j)L_{i+j-2}^{\text{maps}}.$$

- Independent of b, Lie algebras
- [Adler-van Moerbeke]: use matrix integrals

3-constellations

- Constraints $L_i^{\text{const.}} F_{|b=0} = 0$ known from [Fang]
- At $b \neq 0$, we "need" the lemma, so the commutators
- After a lengthy calculation... [VB-Nador]

$$[L_i^{\text{3-const.}}, L_j^{\text{3-const.}}] = 2t(i-j)\sum_{n\geq 1} p_n L_{i+j+n-1}^{\text{3-const.}} + tb(i-j)(i+j)L_{i+j-1}^{\text{3-const.}}$$

$$+ (1+b)t\left(3(i-j)\sum_{n=1}^{\mu-1} + \operatorname{sgn}(i-j)\sum_{n=\mu}^{M-1}(2M-2n-\mu-1)\right)p_n^* L_{i+j-1-n}^{\text{3-const.}}$$

with $M = \max(i,j), \mu = \min(i,j)$

Bipartite maps with controlled degrees

- Bipartite maps with controlled degrees of white vertices, variables q_m
- Set $q_m = 0$ for $m > 3$ [VB-Nador]

$$[L_i^{\mathrm{bip}\leq 3}, L_j^{\mathrm{bip}\leq 3}] = 2t^3(i-j) \sum_{n \geq M-1} J^{(b)}_{i+j-3-n} L_n^{\mathrm{bip}\leq 3} + t^3 b(i-j)(i+j-3) L_{i+j-3}^{\mathrm{bip}\leq 3}$$

$$+ t^3 \operatorname{sgn}(i-j) \sum_{n=\mu-1}^{M-2} (2n - 3\mu + 3) J^{(b)}_{i+j-3-n} L_n^{\mathrm{bip}\leq 3} + t^3 (i-j) \sum_{n=M-1}^{i+j-4} J^{(b)}_{i+j-3-n} L_n^{\mathrm{bip}\leq 3}$$

$$\text{with } J_n^{(b)} = \begin{cases} p_{-n} & \text{if } n < 0 \\ (1+b) n \frac{\partial}{\partial p_n} & \text{if } n > 0 \end{cases}$$

Conclusion

In those models

$$i \frac{\partial H^b}{\partial p_i} = H_i^b = \tilde{H}_i^b$$

Perspectives

- For k-constellations, expect

$$[L_i^{k\text{-const.}}, L_j^{k\text{-const.}}] = \sum_{n_1,\ldots,n_{k-2}} J_{n_1}^{(b)} \cdots J_{n_{k-2}}^{(b)} L_{i+j-1+n_1+\cdots+n_{k-2}}^{k\text{-const.}}$$

- Computer?
- Asymptotics for $g > 0$?
- Motivations from algebraic combinatorics
- Thm [Chapuy-Dołęga]

$$F^b(t,\vec{p},\vec{q},u_1,\ldots,u_k) = \sum_n t^n \sum_\lambda J_\lambda^{(b)}(\vec{p}) J_\lambda^{(b)}(\vec{q}) \frac{1}{j_\lambda^b} \left(\prod_{c=1}^{k} G_\lambda(u_c) \right)$$

  where $J_\lambda^{(b)}(\vec{p})$ is Jack polynomial

- Conj [Goulden-Jackson] for bip maps with controlled face and vertex degrees

$$\sum_n t^n \sum_\lambda J_\lambda^{(b)}(\vec{p}) J_\lambda^{(b)}(\vec{q}) J_\lambda^{(b)}(\vec{r}) \frac{1}{j_\lambda^b} = \sum_n \frac{t^n}{n!} \sum_{\lambda,\mu,\nu} c_{\mu\nu}^\lambda(b) \frac{|\mathcal{C}_\lambda|}{(1+b)^{\ell(\lambda)}} p_\lambda q_\mu r_\nu$$

  with $c_{\mu\nu}^\lambda(b)$ polynomials in b with coefficients in $\mathbb{N}$