How to decide if a D-finite power series is algebraic?

Alin Bostan

Séminaire de Combinatoire Enumérative et Analytique

IHP, 4 April 2024

Theoretical Computer Science 49 (1987) 283–309
North-Holland

AMBIGUITY AND TRANSCENDENCE

Philippe FLAJOLET
INRIA
Rocquencourt
78150 Le Chesnay (France)

ANALYTIC MODELS AND AMBIGUITY OF CONTEXT-FREE LANGUAGES*

Philippe FLAJOLET
INRIA, Rocquencourt, 78150 Le Chesnay Cedex, France

ALGEBRAICALLY INDEPENDENT FORMAL POWER SERIES : A LANGUAGE THEORY INTERPRETATION

J.-P. ALLOUCHE*, P. FLAJOLET**, M. MENDES FRANCE*

* C.N.R.S., U.A. 226 et U.E.R. Mathématiques et Informatique,
Université Bordeaux I, F-33405 TALENCE Cedex.

** INRIA, Domaine de Voluceau, Rocquencourt, B.P. 105, F-78153 LE CHESNAY Cedex.

Goal, motivation, examples

Algebraic and transcendental power series

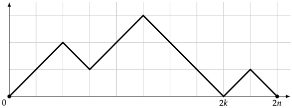
▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P \in \mathbb{Q}[x, y] \setminus \{0\}$.

Otherwise, f is called *transcendental*.

▷ Examples:

- polynomials in $\mathbb{Q}[t]$
- rational functions R in $\mathbb{Q}(t)$ with no pole at $t = 0$
- all powers R^α for $\alpha \in \mathbb{Q}$ and $R(0) = 1$
- sums and products of algebraic power series are algebraic
- the GF $\sum_{n\geq 0} C_n t^n$ of Dyck walks in $\mathbb{N}^2$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



▷ Def extends to Laurent series $f \in \mathbb{Q}((t))$ and Puiseux series $f \in \overline{\mathbb{Q}}((t^{1/\star}))$

Algebraic and transcendental power series

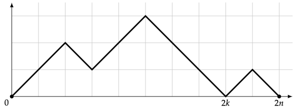
▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P \in \mathbb{Q}[x, y] \setminus \{0\}$.

Otherwise, f is called *transcendental*.

▷ Examples:

- polynomials in $\mathbb{Q}[t]$
- rational functions R in $\mathbb{Q}(t)$ with no pole at $t = 0$
- all powers R^α for $\alpha \in \mathbb{Q}$ and $R(0) = 1$
- sums and products of algebraic power series are algebraic
- the GF $\sum_{n \geq 0} C_n t^n$ of Dyck walks in $\mathbb{N}^2$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



Goal: Given $f \in \mathbb{Q}[[t]]$, either in explicit form (by a formula), or in implicit form (by a functional equation), determine its *algebraicity* or *transcendence*.

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$
- $\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$
- $\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$
- $\sum_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$

Examples (I): power series given explicitly, in closed form

- $\displaystyle\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$
- $\displaystyle\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$
- $\displaystyle\sum_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$
- $\displaystyle\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$
- $\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$
- $\sum_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$
- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$
- $\exp\left(\sum_{n\geq 1} \frac{(2n)!(5n)!^2}{(3n)!^4} \frac{t^n}{n} \right), \quad \exp\left(\sum_{n\geq 1} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \frac{t^n}{n} \right)$

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n$
- $\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$
- $\sum_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$
- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$
- $\exp\left(\sum_{n\geq 1} \frac{(2n)!(5n)!^2}{(3n)!^4} \frac{t^n}{n}\right), \quad \exp\left(\sum_{n\geq 1} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \frac{t^n}{n}\right)$
- $\sum_n (n\text{-th prime number})\, t^n, \quad \sum_n \left(n\text{-th decimal digit of } \sqrt{2}\right) t^n$

▷ Which ones are algebraic?

Examples (I): power series given explicitly, in closed form

- $\sum_n n^{2024} t^n, \quad \sum_n \frac{1}{n^{2024}} t^n,$
- $\sum_n \binom{2n}{n}^{2024} t^n, \quad \sum_n \frac{1}{(2023\,n+1)} \binom{2024\,n}{n} t^n$
- $\sum_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$
- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$
- $\exp\left(\sum_{n\geq 1} \frac{(2n)!(5n)!^2}{(3n)!^4} \frac{t^n}{n}\right), \quad \exp\left(\sum_{n\geq 1} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \frac{t^n}{n}\right)$
- $\sum_n (n\text{-th prime number})\, t^n, \quad \sum_n \left(n\text{-th decimal digit of } \sqrt{2}\right) t^n$

▷ Which ones are algebraic?

Examples (II): power series given implicitly, as solutions of equations

- $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$, the unique solution of

$$t^2(1+t)(1-2t)(1+4t)(1-8t)f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right)f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right)f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right)f(t) = 0,$$

Examples (II): power series given implicitly, as solutions of equations

- $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$, the unique solution of

$$t^2(1+t)(1-2t)(1+4t)(1-8t)\,f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right) f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right) f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right) f(t) = 0,$$

- $f(t) = F(1,t)$ where $F(x,t)$ is the unique solution in $\mathbb{Q}[x][[t]]$ of

$$F(x,t) = 1 + tx^2\, F(x,t)^2 + tx\, \frac{xF(x,t) - F(1,t)}{x-1},$$

Examples (II): power series given implicitly, as solutions of equations

- $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$, the unique solution of

$$t^2(1+t)(1-2t)(1+4t)(1-8t)\,f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right) f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right) f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right) f(t) = 0,$$

- $f(t) = F(1,t)$ where $F(x,t)$ is the unique solution in $\mathbb{Q}[x][[t]]$ of

$$F(x,t) = 1 + tx^2\,F(x,t)^2 + tx\,\frac{xF(x,t) - F(1,t)}{x-1},$$

- $f(t) = F(1,1,t)$ where $F(x,y,t)$ is the unique solution in $\mathbb{Q}[x,y][[t]]$ of

$$F(x,y,t) = 1 + tyF(x,y,t) + tx\,\frac{F(x,y,t) - F(x,0,t)}{y} + t\,\frac{F(x,y,t) - F(0,y,t)}{x}.$$

▷ Which ones are algebraic?

Examples (II): power series given implicitly, as solutions of equations

- $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$, the unique solution of

$$t^2(1+t)(1-2t)(1+4t)(1-8t)\,f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right) f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right) f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right) f(t) = 0,$$

- $f(t) = F(1,t)$ where $F(x,t)$ is the unique solution in $\mathbb{Q}[x][[t]]$ of

$$F(x,t) = 1 + tx^2\,F(x,t)^2 + tx\,\frac{xF(x,t) - F(1,t)}{x-1},$$

- $f(t) = F(1,1,t)$ where $F(x,y,t)$ is the unique solution in $\mathbb{Q}[x,y][[t]]$ of

$$F(x,y,t) = 1 + tyF(x,y,t) + tx\,\frac{F(x,y,t) - F(x,0,t)}{y} + t\,\frac{F(x,y,t) - F(0,y,t)}{x}.$$

▷ Which ones are algebraic?

D-finite power series

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *D-finite* (*differentially finite*) if it is a solution of some LDE (i.e., *linear* ODE)

$$c_r(t)f^{(r)}(t) + \cdots + c_0(t)f(t) = 0$$

for some $c_i \in \mathbb{Q}(t)$, with c_r nonzero. (r is called the *order* of this LDE.)

Europ. J. Combinatorics (1980) **1**, 175–188

Differentiably Finite Power Series

R. P. STANLEY*

A formal power series $\sum f(n)x^n$ is said to be differentiably finite if it satisfies a linear differential equation with polynomial coefficients. Such power series arise in a wide variety of problems in enumerative combinatorics. The basic properties of such series of significance to combinatorics are surveyed. Some reciprocity theorems are proved which link two such series together. A number of examples, applications and open problems are discussed.

1. INTRODUCTION

Recently there has been interest [2], [3], [16] in the problem of computing quickly the coefficients of a power series $F(x) = \sum_{n\geqslant 0} f(n)x^n$, where say $F(x)$ is defined by a functional equation or as a function of other power series. If the coefficients $f(n)$ have a combinatorial meaning, then a fast algorithm for computing $f(n)$ would also be of combinatorial interest. Here we consider a class of power series, which we call *differentiably finite* (or D-finite, for short), whose coefficients can be quickly computed in a simple way. We consider various operations on power series which preserve the property of being D-finite, and give examples of operations which don't preserve this property. We mention some classes of power series for which it seems quite difficult to decide whether they are D-finite. Everything we say can be extended routinely from power series to Laurent series having finitely many terms with negative exponents, though for simplicity we will restrict ourselves to power series. Moreover, we will consider only complex coefficients, though virtually all of what we do is valid over any field of characteristic zero (and much is valid over any field).

The class of D-finite power series has been subject to extensive investigation, particularly within the theory of differential equations. However, a systematic exposition of their properties from a combinatorial point of view seems not to have been given before. Many of our results can therefore be found scattered throughout the literature, so this paper should be regarded as about 75% expository. To simplify and unify the concepts and proofs we have used the terminology and elementary theory of linear algebra, though all explicit dependence on linear algebra could be avoided without great difficulty.

Let us now turn to the basic definition of this paper. First note that the field $\mathbb{C}((x))$ of all formal Laurent series over $\mathbb{C}$ of the form $\sum_{n\geqslant n_0} f(n)x^n$ for some $n_0 \in \mathbb{Z}$ contains the field $\mathbb{C}(x)$ of rational functions of x, and $\mathbb{C}((x))$ has the structure of a vector space over $\mathbb{C}(x)$.

DEFINITION 1.1. A formal power series $y \in \mathbb{C}[[x]]$ is said to be *differentiably finite* (or D-finite) if y together with all its derivatives $y^{(n)} = d^n y/dx^n$, $n \geqslant 1$, span a finite-dimensional subspace of $\mathbb{C}((x))$, regarded as a vector space over the field $\mathbb{C}(x)$.

THEOREM 1.2. *The following three conditions on a formal power series $y \in \mathbb{C}[[x]]$ are equivalent.*

(i) *y is D-finite.*

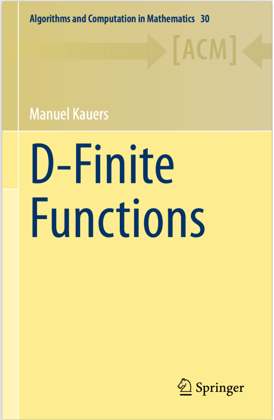
(ii) *There exist finitely many polynomials $q_0(x), \ldots, q_k(x)$, not all 0, and a polynomial $q(x)$, such that*

$$q_k(x)y^{(k)} + \cdots + q_1(x)y' + q_0(x)y = q(x). \quad (1)$$

* Partially supported by the National Science Foundation.

175

0195-6698/80/020175+14 $01.00/0 © 1980 Academic Press Inc. (London) Limited



D-finite power series

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *D-finite* (*differentially finite*) if it is a solution of some LDE (i.e., *linear* ODE)

$$c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$$

for some $c_i \in \mathbb{Q}(t)$, with c_r nonzero. (r is called the *order* of this LDE.)

▷ Examples:

- $\exp(t) := \sum_{n \geq 0} t^n / n!$, solution of $f'(t) = f(t)$
- $\log(1-t) := -\sum_{n \geq 1} t^n / n$, solution of $(t-1) f''(t) + f'(t) = 0$
- $\sqrt[N]{R(t)}$ for $R \in \mathbb{Q}(t)$, solution of $f'(t)/f(t) = \frac{1}{N} R'(t)/R(t)$
- any algebraic power series ("*Abel's theorem*")
- $\arctan(t)$, solution of $(t^2+1) f''(t) + 2t f'(t) = 0$, but not $\tan(t)$
- sums and products of D-finite are D-finite

D-finite power series

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *D-finite* (*differentially finite*) if it is a solution of some LDE (i.e., *linear* ODE)

$$c_r(t)f^{(r)}(t) + \cdots + c_0(t)f(t) = 0$$

for some $c_i \in \mathbb{Q}(t)$, with c_r nonzero. (r is called the *order* of this LDE.)

▷ Examples:

- $\exp(t) := \sum_{n\geq 0} t^n/n!$, solution of $f'(t) = f(t)$
- $\log(1-t) := -\sum_{n\geq 1} t^n/n$, solution of $(t-1)f''(t) + f'(t) = 0$
- $\sqrt[N]{R(t)}$ for $R \in \mathbb{Q}(t)$, solution of $f'(t)/f(t) = \frac{1}{N}R'(t)/R(t)$
- any algebraic power series ("*Abel's theorem*")
- $\arctan(t)$, solution of $(t^2+1)f''(t) + 2tf'(t) = 0$, but not $\tan(t)$
- sums and products of D-finite are D-finite

▷ Simple but important property: $\sum_{n\geq 0} a_n t^n$ is D-finite if and only if $(a_n)_{n\geq 0}$ is *P-finite* (i.e., it satisfies a linear recurrence with coefficients in $\mathbb{Q}[n]$)

Linear differential operators

• $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$

Linear differential operators

- $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$
- Usual $+$; *skew* multiplication $\star$ defined by $\partial_t \star R(t) = R(t) \star \partial_t + R'(t)$

Linear differential operators

- $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$
- Usual $+$; *skew* multiplication $\star$ defined by $\partial_t \star R(t) = R(t) \star \partial_t + R'(t)$
- Commutation rule models Leibniz's rule $(RS)' = RS' + R'S$

- $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$
- Usual $+$; *skew* multiplication $\star$ defined by $\partial_t \star R(t) = R(t) \star \partial_t + R'(t)$
- Commutation rule models Leibniz's rule $(RS)' = RS' + R'S$

$\longrightarrow$ algebraic formalization of the notion of LDE

$$c_r(t)y^{(r)}(t) + \cdots + c_1(t)y'(t) + c_0(t)y(t) = 0$$

$$\Longleftrightarrow$$

$$\mathscr{L}(y) = 0, \quad \text{where} \quad \mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$$

Linear differential operators

- $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$
- Usual $+$; *skew* multiplication $\star$ defined by $\partial_t \star R(t) = R(t) \star \partial_t + R'(t)$
- Commutation rule models Leibniz's rule $(RS)' = RS' + R'S$

$\longrightarrow$ algebraic formalization of the notion of LDE

$$c_r(t)y^{(r)}(t) + \cdots + c_1(t)y'(t) + c_0(t)y(t) = 0$$

$$\Longleftrightarrow$$

$$\mathscr{L}(y) = 0, \quad \text{where} \quad \mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$$

$\triangleright$ If $c_r \neq 0$, then $r = \deg_{\partial_t}(\mathscr{L})$ is called the **order** of $\mathscr{L}$, denoted $\operatorname{ord}(\mathscr{L})$

Linear differential operators

- $\mathbb{Q}(t)\langle\partial_t\rangle$ = the (non-commutative) algebra of linear differential operators ("skew polynomials") $\mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$ with $c_i \in \mathbb{Q}(t)$
- Usual $+$; *skew* multiplication $\star$ defined by $\partial_t \star R(t) = R(t) \star \partial_t + R'(t)$
- Commutation rule models Leibniz's rule $(RS)' = RS' + R'S$

$\longrightarrow$ algebraic formalization of the notion of LDE

$$c_r(t)y^{(r)}(t) + \cdots + c_1(t)y'(t) + c_0(t)y(t) = 0$$
$$\Longleftrightarrow$$
$$\mathscr{L}(y) = 0, \quad \text{where} \quad \mathscr{L} = c_r(t)\partial_t^r + \cdots + c_1(t)\partial_t + c_0(t)$$

▷ If $c_r \neq 0$, then $r = \deg_{\partial_t}(\mathscr{L})$ is called the **order** of $\mathscr{L}$, denoted $\operatorname{ord}(\mathscr{L})$

Theorem [Libri 1833; Brassinne 1864; Wedderburn 1932; Ore 1932]

$\mathbb{Q}(t)\langle\partial_t\rangle$ is a non-commutative (right) Euclidean domain: for $\mathscr{A} \in \mathbb{Q}(t)\langle\partial_t\rangle$ and $\mathscr{B} \in \mathbb{Q}(t)\langle\partial_t\rangle \setminus \{0\}$, there exist unique $\mathscr{Q}, \mathscr{R} \in \mathbb{Q}(t)\langle\partial_t\rangle$ such that

$$\mathscr{A} = \mathscr{Q}\mathscr{B} + \mathscr{R} \qquad \text{and} \qquad \operatorname{ord}(\mathscr{R}) < \operatorname{ord}(\mathscr{B}).$$

(This is called the Euclidean right division of $\mathscr{A}$ by $\mathscr{B}$.)

Main question today: *How to decide if a D-finite power series is algebraic?*

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

Goal: Given a D-finite $f \in \mathbb{Q}[[t]]$, by a linear differential equation and enough initial terms, determine its *algebraicity* or *transcendence*.

▷ Example: What is the nature of $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$ such that

$$t^2 (1+t)(1-2t)(1+4t)(1-8t) f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right) f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right) f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right) f(t) = 0?$$

Main question today: *How to decide if a D-finite power series is algebraic?*

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

Equivalent goal: Given a P-finite sequence of rational numbers $(a_n)_{n\geq 0}$ by a linear recurrence and enough initial terms, determine the *algebraicity* or the *transcendence* of its generating function $\sum_{n\geq 0} a_n t^n$.

▷ Example: What is the nature of $f(t) = \sum_{n\geq 0} a_n t^n$, where $(a_n)_{n\geq 0}$ is defined by $a_0 = 1, a_1 = 3, a_2 = 18, a_3 = 105$ and

$$(n+4)(n+5)^2 a_{n+4} - (n+4)\left(5n^2+43n+96\right) a_{n+3} - 6(5n+22)(n+4)(n+3) a_{n+2}$$
$$+8(n+2)\left(5n^2+15n+1\right) a_{n+1} + 64(n+3)(n+2)(n+1) a_n = 0?$$

▷ NB: Integrality and algebraicity are related; deciding integrality is harder!

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

E.g.,

$$f = \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order LDE

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

▷ An algorithm should recognize (from this data) that f is transcendental.

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

▷ Notation: For a D-finite series f, we write $\mathscr{L}_f^{\min}$ for the least-order, monic, linear differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

▷ Notation: For a D-finite series f, we write $\mathscr{L}_f^{\min}$ for the least-order, monic, linear differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

▷ Warning: $\mathscr{L}_f^{\min}$ is not known a priori; only some multiple $\mathscr{L}$ of it is given.

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

▷ Notation: For a D-finite series f, we write $\mathscr{L}_f^{\min}$ for the least-order, monic, linear differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

▷ Warning: $\mathscr{L}_f^{\min}$ is not known a priori; only some multiple $\mathscr{L}$ of it is given.

▷ Difficulty: $\mathscr{L}_f^{\min}$ might not be irreducible. E.g., $\mathscr{L}_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.

Related problems

$$\mathscr{L}(y(t)) := c_r(t)y^{(r)}(t) + \cdots + c_0(t)y(t) = 0$$

- **(S)** *Stanley's problem*: Decide if a given solution f of $\mathscr{L}(y) = 0$ is algebraic
- **(F)** *Fuchs' problem*: Decide if all solutions of $\mathscr{L}(y) = 0$ are algebraic
- **(L)** *Liouville's problem*: Decide if $\mathscr{L}(y) = 0$ has at least one algebraic solution ($\neq 0$)

Related problems

$$\mathscr{L}(y(t)) := c_r(t)y^{(r)}(t) + \cdots + c_0(t)y(t) = 0$$

(S) *Stanley's problem*: Decide if a given solution f of $\mathscr{L}(y) = 0$ is algebraic

(F) *Fuchs' problem*: Decide if all solutions of $\mathscr{L}(y) = 0$ are algebraic

(L) *Liouville's problem*: Decide if $\mathscr{L}(y) = 0$ has at least one algebraic solution ($\neq 0$)

▷ When $\mathscr{L}$ is irreducible, problems **(S)**, **(F)** and **(L)** are equivalent

Related problems

$$\mathscr{L}(y(t)) := c_r(t)y^{(r)}(t) + \cdots + c_0(t)y(t) = 0$$

(S) *Stanley's problem*: Decide if a given solution f of $\mathscr{L}(y) = 0$ is algebraic

(F) *Fuchs' problem*: Decide if all solutions of $\mathscr{L}(y) = 0$ are algebraic

(L) *Liouville's problem*: Decide if $\mathscr{L}(y) = 0$ has at least one algebraic solution ($\neq 0$)

▷ When $\mathscr{L}$ is irreducible, problems **(S)**, **(F)** and **(L)** are equivalent

Today's main results: how to solve **(S), (F)** and **(L)** for arbitrary $\mathscr{L}$

Motivations

- Number theory: a first step towards proving the transcendence of a complex number is proving that some power series is transcendental
- Combinatorics: the nature of generating functions may reveal strong underlying structures
- Computer science: are algebraic power series (intrinsically) easier to manipulate?

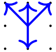
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

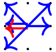
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

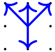
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

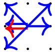
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots,$$

where $a_n = \# \{$  $-$ walks of length n in $\mathbb{N}^2$ starting at $(0,0) \}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots,$$

where $a_n = \# \{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star, 0) \}$

Question: *What is the nature of these three power series?*

Transcendence criteria

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

- Algebraic properties

  f is D-finite and $\mathscr{L}_f^{\min}$ has only algebraic solutions [Abel, 1827; Tannery, 1875]

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

- Algebraic properties

  f is D-finite and $\mathscr{L}_f^{\min}$ has only algebraic solutions [Abel, 1827; Tannery, 1875]

- Arithmetic properties
  - f is globally bounded: $\exists C \in \mathbb{N}^*$ with $a_n C^n \in \mathbb{Z}$ for $n \geq 1$ [Eisenstein, 1852]

    In particular, denominators of a_n's have finitely many prime divisors
  - $\partial_t^p \bmod \mathscr{L}_f^{\min} = 0 \pmod{p}$ for primes $p \gg 0$ "Cartier's Lemma" [Katz, 1970]

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

- Algebraic properties

  f is D-finite and $\mathscr{L}_f^{\min}$ has only algebraic solutions [Abel, 1827; Tannery, 1875]

- Arithmetic properties
  - f is globally bounded: $\exists C \in \mathbb{N}^*$ with $a_n C^n \in \mathbb{Z}$ for $n \geq 1$ [Eisenstein, 1852]

    In particular, denominators of a_n's have finitely many prime divisors
  - $\partial_t^p \bmod \mathscr{L}_f^{\min} = 0 \pmod p$ for primes $p \gg 0$ "Cartier's Lemma" [Katz, 1970]

- Analytic properties$^{(\star)}$
  - $f(t)$ has finite nonzero radius of convergence
  - $(a_n)_n$ has "nice" asymptotics [Puiseux, 1850; Darboux, 1878; Flajolet, 1987]

  Typically, $a_n \sim \kappa \, \rho^n \, n^\alpha$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and $\rho \in \overline{\mathbb{Q}}$ and $\kappa \cdot \underbrace{\Gamma(\alpha+1)}_{:= \int_0^\infty t^\alpha e^{-t} \, \mathrm{d}t} \in \overline{\mathbb{Q}}$

$^{(\star)}$ "It is usually 'easy' to establish transcendence of functions, by exhibiting a local expansion that contradicts the Newton–Puiseux Theorem" [Flajolet, Sedgewick, 2009]

…and the resulting transcendence criteria

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- f is not D-finite $\qquad \prod_{n\geq 1} \frac{1}{1-t^n}$
- f has infinitely many primes in the denominators $\qquad \sum_{n\geq 1} \frac{1}{n} t^n$
- $(a_n)_n$ has incompatible asymptotics $\qquad \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$ (†)
- $\partial_t^p \bmod \mathscr{L}_f^{\min} \neq 0 \pmod p$ for infinitely many primes p $\qquad \exp(t)$

then f is transcendental

(†) $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ and $\frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \notin \overline{\mathbb{Q}}$

…and the resulting transcendence criteria

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- f is not D-finite $\qquad \prod_{n\geq 1} \frac{1}{1-t^n}$
- f has infinitely many primes in the denominators $\qquad \sum_{n\geq 1} \frac{1}{n} t^n$
- $(a_n)_n$ has incompatible asymptotics $\qquad \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$ (†)
- $\partial_t^p \bmod \mathscr{L}_f^{\min} \neq 0 \pmod{p}$ for infinitely many primes p $\qquad \exp(t)$

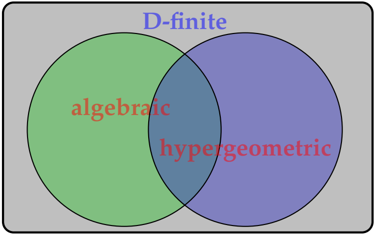
then f is transcendental

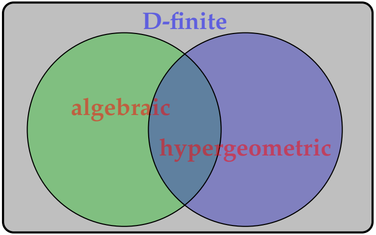
▷ The *Grothendieck-Katz conjecture* predicts last criterion is an equivalence (!)

(†) $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ and $\frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \notin \overline{\mathbb{Q}}$

Hypergeometric case

Hypergeometric series

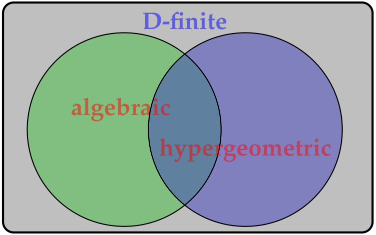




$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P\big(t, f(t)\big) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$

Hypergeometric series

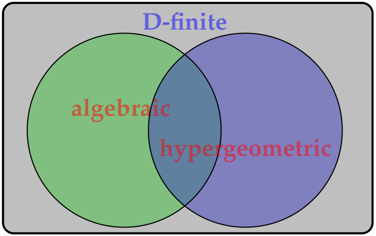


$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

Hypergeometric series



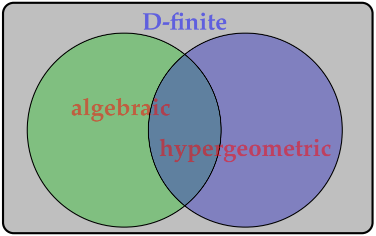
$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]]$ is

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., $\ln(1-t)$; $\frac{\arcsin(\sqrt{t})}{\sqrt{t}}$; $(1-t)^{\alpha}$, $\alpha \in \mathbb{Q}$

Hypergeometric series



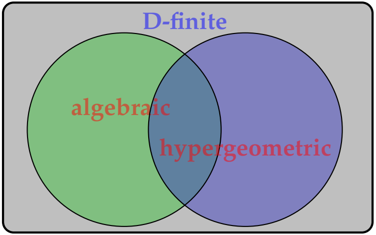
$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., ${}_2F_1\left(\begin{matrix}\alpha\ \beta \\ \gamma\end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{t^n}{n!}$, $\quad (\delta)_n = \prod_{\ell=0}^{n-1} (\delta + \ell)$

Hypergeometric series



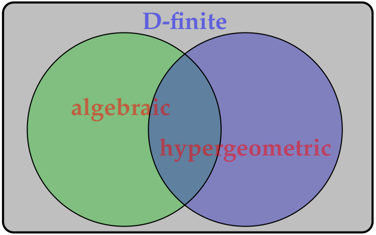
$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., ${}_pF_q\left(\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{t^n}{n!}$

Hypergeometric series



$$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., ${}_pF_q\left(\begin{matrix} \alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q \end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{t^n}{n!}$

Theorem [Schwarz 1873; Landau 1904, 1911; Stridsberg 1911; Errera 1913; Katz 1972; Christol 1986; Beukers, Heckman 1989; Katz 1990; Fürnsinn, Yurkevich 2024]

Full characterization of { *hypergeom* } ∩ { *algebraic* }

Algebraic hypergeometric series

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \middle| \, t\right)$ is algebraic iff $\{e^{2\pi i r a_j}, j \leq k\}$ and $\{e^{2\pi i r b_\ell}, \ell \leq k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then

$${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \,\middle|\, t\right)$$

is algebraic iff $\{e^{2\pi i r a_j}, j \leq k\}$ and $\{e^{2\pi i r b_\ell}, \ell \leq k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.

Groupe d'Etude d'Analyse ultramétrique. (1986/87)
N° 8 , 16 pages.

le 15 décembre 1986

Exposé n° 8

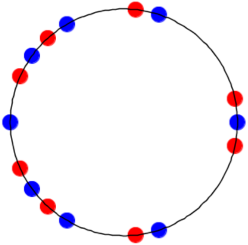
FONCTIONS HYPERGEOMETRIQUES BORNEES

GILLES CHRISTOL

PROPOSITION 3 : *Toute fonction hypergéométrique F réduite et de hauteur 1 est globalement bornée si et seulement si, pour tout Δ tel que $(\Delta, N) = 1$, les nombres $\exp(2i\pi\Delta a_i)$ et $\exp(2i\pi\Delta b_i)$ sont entrelacés sur le cercle unité.*

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \,\middle|\, t\right)$ is algebraic iff $\{e^{2\pi i r a_j}, j \leq k\}$ and $\{e^{2\pi i r b_\ell}, \ell \leq k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.

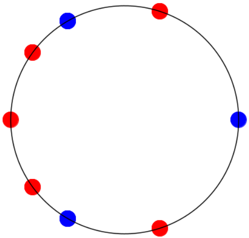


▷ $\displaystyle\sum_n \frac{(30n)!\,n!}{(15n)!\,(10n)!\,(6n)!} t^n = {}_8F_7\left(\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{5} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{4}{5} & \end{matrix} \,\middle|\, 2^{14}\, 3^9\, 5^5\, t\right)$ is algebraic

Algebraic hypergeometric series

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

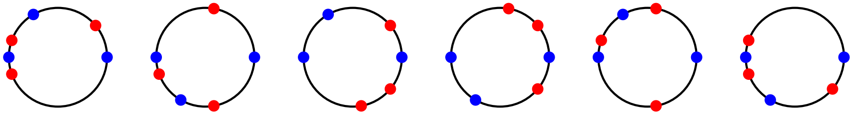
Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then
$${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \,\middle|\, t\right)$$
is algebraic iff $\{e^{2\pi i r a_j}, j \le k\}$ and $\{e^{2\pi i r b_\ell}, \ell \le k\}$ interlace on the unit circle for all $1 \le r < D$ with $\gcd(r, D) = 1$.



▷ $$\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n = {}_9F_8\left(\begin{matrix} \frac15 & \frac15 & \frac25 & \frac25 & \frac12 & \frac35 & \frac35 & \frac45 & \frac45 \\ \frac13 & \frac13 & \frac13 & \frac13 & \frac23 & \frac23 & \frac23 & \frac23 & \end{matrix} \,\middle|\, \frac{2^2\, 5^{10}}{3^{12}}\, t\right)$$ is transcendental

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

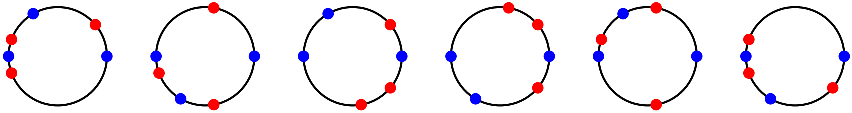
Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \middle| \, t\right)$ is algebraic iff $\{e^{2\pi i r a_j}, j \leq k\}$ and $\{e^{2\pi i r b_\ell}, \ell \leq k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.



▷ ${}_3F_2\left(\begin{matrix} \frac{1}{9} & \frac{4}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{3} & \end{matrix} \middle| \, 3^6\, t\right) = 1 + 120\, t + 54600\, t^2 + 29995680\, t^3 + 17853428736\, t^4 + 11111241596928\, t^5 + 7114982545305600\, t^6 + \cdots$ is transcendental

Theorem [Beukers, Heckman, 1989] ("interlacing criterion")

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \,\middle|\, t\right)$ is algebraic iff $\{e^{2\pi i r a_j}, j \le k\}$ and $\{e^{2\pi i r b_\ell}, \ell \le k\}$ interlace on the unit circle for all $1 \le r < D$ with $\gcd(r, D) = 1$.



▷ ${}_3F_2\left(\begin{matrix} \frac{1}{9} & \frac{4}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{3} & \end{matrix} \,\middle|\, 3^6\, t\right) = 1 + 120\, t + 54600\, t^2 + 29995680\, t^3 + 17853428736\, t^4 + 11111241596928\, t^5 + 7114982545305600\, t^6 + \frac{60411016459487232000}{13}\, t^7 + \cdots$ is transcendental

Algebraicity of hypergeometric series with arbitrary parameters

Theorem [Fürnsinn, Yurkevich, 2024]

A hypergeometric series $F = {}_pF_q \in \mathbb{Q}[[t]] \setminus \mathbb{Q}[t]$ is algebraic if and only if its contraction F^c has parameters in $\mathbb{Q}$ and satisfies the interlacing criterion.

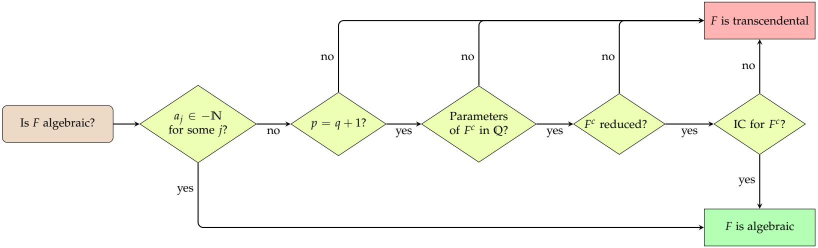
(F^c is obtained from F by removing all pairs (a_j, b_ℓ) with $a_j - b_\ell \in \mathbb{N}$.)

Algebraicity of hypergeometric series with arbitrary parameters

Theorem [Fürnsinn, Yurkevich, 2024]

A hypergeometric series $F = {}_pF_q \in \mathbb{Q}[[t]] \setminus \mathbb{Q}[t]$ is algebraic if and only if its contraction F^c has parameters in $\mathbb{Q}$ and satisfies the interlacing criterion.

(F^c is obtained from F by removing all pairs (a_j, b_ℓ) with $a_j - b_\ell \in \mathbb{N}$.)



Stanley's problem

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

[Stanley, 1980]

E.g.,

$$f = \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order equation

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

▷ An algorithm should recognize (from this data) that f is transcendental.

A few starting remarks on Stanley's problem

▷ Analogy between transcendence in $\mathbb{Q}[[t]]$ and irreducibility in $\mathbb{Q}[t]$:

- "generic" series are transcendent, "generic" polynomials are irreducible
- sufficient criteria exist (e.g., Eisenstein's), but none is also necessary
- irreducibility is decidable; what about transcendence?

A few starting remarks on Stanley's problem

▷ Analogy between transcendence in $\mathbb{Q}[[t]]$ and irreducibility in $\mathbb{Q}[t]$:

- "generic" series are transcendent, "generic" polynomials are irreducible
- sufficient criteria exist (e.g., Eisenstein's), but none is also necessary
- irreducibility is decidable; what about transcendence?

▷ The minimal polynomial can have arbitrarily large size (degrees) w.r.t. the size (order/degree) of the differential equation:

solution of $N(t-1)f'(t) - f(t) = 0$, $f(0) = 1$ satisfies $f^N = 1 - t$

A few starting remarks on Stanley's problem

▷ Analogy between transcendence in $\mathbb{Q}[[t]]$ and irreducibility in $\mathbb{Q}[t]$:

- "generic" series are transcendent, "generic" polynomials are irreducible
- sufficient criteria exist (e.g., Eisenstein's), but none is also necessary
- irreducibility is decidable; what about transcendence?

▷ The minimal polynomial can have arbitrarily large size (degrees) w.r.t. the size (order/degree) of the differential equation:

solution of $N(t-1)f'(t) - f(t) = 0,\ f(0) = 1$ satisfies $f^N = 1 - t$

▷ No characterization for coefficient sequences of algebraic power series

- larger class: *D-finite functions* $\Longleftrightarrow$ *P-finite sequences*
- smaller class: *rational functions* $\Longleftrightarrow$ *C-finite sequences*
- *diagonals* $\overset{\text{Christol's}}{\underset{\text{conjecture}}{\Longleftrightarrow}}$ *P-finite, almost integer, seq. with geometric growth*

  (NB: in positive characteristic p, *algebraic functions* $\Longleftrightarrow$ *p-automatic sequences*)

A bit of history

▷ [Liouville, 1833]: algorithm for (basis of) *rational solutions* of LDEs
⟶ solves the rational versions $\mathbf{(S_{rat})}$, $\mathbf{(F_{rat})}$ and $\mathbf{(L_{rat})}$ of **(S)**, **(F)** and **(L)**

▷ [Fuchs, 1866]: characterization of LDEs having *only rational solutions*
⟶ alternative solution to $\mathbf{(F_{rat})}$

▷ [Liouville, 1833]: algorithm for (basis of) *rational solutions* of LDEs
⟶ solves the rational versions **(S_{rat})**, **(F_{rat})** and **(L_{rat})** of **(S)**, **(F)** and **(L)**

▷ [Fuchs, 1866]: characterization of LDEs having *only rational solutions*
⟶ alternative solution to **(F_{rat})**

▷ [Schwarz, 1873]: *solution to* **(F)** *for second order LDEs with 3 singular points*
(Gauss hypergeometric equation $t(t-1)y'' + ((a+b+1)t-c)y' + aby = 0$)

▷ [Baldassarri & Dwork 1979]: *solution to* **(F)** *for arbitrary second order LDEs*, building on works by [Klein, 1878] and [Fuchs, 1878]

▷ [Singer, 1979]: *full solution to* **(F)** building on works by [Jordan, 1880], [Painlevé, 1887], [Boulanger, 1898] and [Risch, 1969]

▷ [Katz, 1972, 1982], [André, 2004]: *Grothendieck–Katz p-curvature conjecture*: local-global principle for LDEs, (conjectural) arithmetic solution to **(F)**

▷ [Liouville, 1833]: algorithm for (basis of) *rational solutions* of LDEs
⟶ solves the rational versions $(\mathbf{S}_{\text{rat}})$, $(\mathbf{F}_{\text{rat}})$ and $(\mathbf{L}_{\text{rat}})$ of **(S)**, **(F)** and **(L)**

▷ [Fuchs, 1866]: characterization of LDEs having *only rational solutions*
⟶ alternative solution to $(\mathbf{F}_{\text{rat}})$

▷ [Schwarz, 1873]: *solution to* **(F)** *for second order LDEs with 3 singular points* (Gauss hypergeometric equation $t(t-1)y'' + ((a+b+1)t-c)y' + aby = 0$)

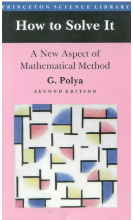
▷ [Baldassarri & Dwork 1979]: *solution to* **(F)** *for arbitrary second order LDEs*, building on works by [Klein, 1878] and [Fuchs, 1878]

▷ [Singer, 1979]: *full solution to* **(F)** building on works by [Jordan, 1880], [Painlevé, 1887], [Boulanger, 1898] and [Risch, 1969]

▷ [Katz, 1972, 1982], [André, 2004]: *Grothendieck–Katz p-curvature conjecture*: local-global principle for LDEs, (conjectural) arithmetic solution to **(F)**

▷ Many tools: geometry (Schwarz, Klein), invariant theory (Fuchs, Gordan), group theory (Jordan), diff. Galois theory (Vessiot, Singer, Hrushovski), number theory and algebraic geometry (Grothendieck, Katz, André)

Guess-and-Prove



Guessing and Proving

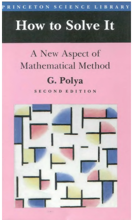
George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.



Guessing and Proving

George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

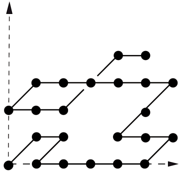
First guess, then prove.

generate data → make conjectures → prove them

- $g(i,j,n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

▷ **Question**: What is the nature of the generating function

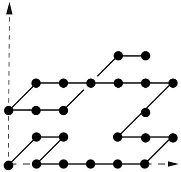
$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \ ?$$



- $g(i,j,n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

▷ **Question**: What is the nature of the generating function

$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \ ?$$



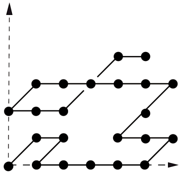
▷ **Algebraic reformulation**: Solve the "kernel equation"

$$G(x,y,t) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right)G(x,y,t) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right)G(0,y,t) - t\,\frac{1}{xy}\Big(G(x,0,t) - G(0,0,t)\Big)$$

- $g(i,j,n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

▷ **Question**: What is the nature of the generating function

$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \ ?$$



Answer: [B., Kauers, 2010] $G(x,y,t)$ is an algebraic function†.

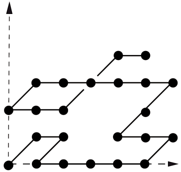
▷ **Approach**:

1. Generate data: compute $G(x,y,t)$ to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0,t)$ and $G(0,y,t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bit digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y,t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (6 DVDs!)

Guess-and-Prove for Gessel walks

- $g(i,j,n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)



▷ **Question**: What is the nature of the generating function

$$G(x,y,t) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, x^i y^j t^n \text{ ?}$$

Answer: [B., Kauers, 2010] $G(x,y,t)$ is an algebraic function†.

▷ **Approach**: ⟶ very general and robust!

1. Generate data: compute $G(x,y,t)$ to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0,t)$ and $G(0,y,t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bit digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y,t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx$ 30 Gb (6 DVDs!)

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)\ 3\,t^2\,(16\,t-1)g'''(t) + 2\,t\,(128\,t-7)\,g''(t) + 2\,(122\,t-5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)$ $3\,t^2\,(16\,t - 1)g'''(t) + 2\,t\,(128\,t - 7)\,g''(t) + 2\,(122\,t - 5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2t + 11t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)$ $3t^2(16t-1)g'''(t) + 2t(128t-7)g''(t) + 2(122t-5)g'(t) + 20g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)\; 3\,t^2\,(16\,t-1) g'''(t) + 2\,t\,(128\,t-7)\,g''(t) + 2\,(122\,t-5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists$ root $r(t) \in \mathbb{Q}[[t]]$ of P with $r(0) = 1$.

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)\ 3\,t^2\,(16\,t - 1)g'''(t) + 2\,t\,(128\,t - 7)\,g''(t) + 2\,(122\,t - 5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists$ root $r(t) \in \mathbb{Q}[[t]]$ of P with $r(0) = 1$.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite and satisfies $(\star) \implies$ since $r(t) = 1 + 2\,t + 11\,t^2 + \cdots$, by uniqueness $g = r$, hence g is algebraic.

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)\ 3\,t^2\,(16\,t-1)g'''(t) + 2\,t\,(128\,t-7)\,g''(t) + 2\,(122\,t-5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t,g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists$ root $r(t) \in \mathbb{Q}[[t]]$ of P with $r(0) = 1$.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite and satisfies $(\star) \Longrightarrow$ since $r(t) = 1 + 2\,t + 11\,t^2 + \cdots$, by uniqueness $g = r$, hence g is algebraic.

```
> gs:=op(-1, dsolve(deqg, g(t), series, order = 100)):
> P:=gfun:-seriestoalgeq(gs, g(t))[1]:
> gfun:-diffeqtohomdiffeq(gfun:-algeqtodiffeq(P, g(t), g(t)), g(t));
```

An easier, but typical Guess-and-Prove algebraicity proof

Theorem ["Gessel excursions are algebraic"]

The unique solution $g(t) = 1 + 2\,t + 11\,t^2 + \cdots$ in $\mathbb{Q}[[t]]$ of
$(\star)\ 3\,t^2\,(16\,t-1)g'''(t) + 2\,t\,(128\,t-7)\,g''(t) + 2\,(122\,t-5)\,g'(t) + 20\,g(t) = 0$
is algebraic.

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists$ root $r(t) \in \mathbb{Q}[[t]]$ of P with $r(0) = 1$.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite and satisfies $(\star) \implies$ since $r(t) = 1 + 2\,t + 11\,t^2 + \cdots$, by uniqueness $g = r$, hence g is algebraic.

```
> gs:=op(-1, dsolve(deqg, g(t), series, order = 100)):
> P:=gfun:-seriestoalgeq(gs, g(t))[1]:
> gfun:-diffeqtohomdiffeq(gfun:-algeqtodiffeq(P, g(t), g(t)), g(t));
```

▷ The approach applies (in principle) to any instance of Stanley's problem.

Singer's algorithm
and
Stanley's problem

Singer's algorithm

Problem **(F)**: Decide if *all solutions* of a given LDE $\mathscr{L}$ of order r *are algebraic*

• Starting point [Jordan, 1878]: If so, then for some solution y of $\mathscr{L}$, $u = y'/y$ has alg. degree at most $(49r)^{r^2}$ and satisfies a Riccati equation of order $r-1$

Algorithm ($\mathscr{L}$ irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

1. Decide if the Riccati equation has an algebraic solution u of degree at most $(49r)^{r^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

Singer's algorithm

Problem **(F)**: Decide if *all solutions* of a given LDE $\mathscr{L}$ of order r *are algebraic*

• Starting point [Jordan, 1878]: If so, then for some solution y of $\mathscr{L}$, $u = y'/y$ has alg. degree at most $(49r)^{r^2}$ and satisfies a Riccati equation of order $r-1$

Algorithm ($\mathscr{L}$ irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

1. Decide if the Riccati equation has an algebraic solution u of degree at most $(49r)^{r^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

▷ [Singer, 1979]: generalization to any input $\mathscr{L}$ $\longrightarrow$ requires LDE factoring

Problem **(F)**: Decide if *all solutions* of a given LDE $\mathscr{L}$ of order r *are algebraic*

• Starting point [Jordan, 1878]: If so, then for some solution y of $\mathscr{L}$, $u = y'/y$ has alg. degree at most $(49r)^{r^2}$ and satisfies a Riccati equation of order $r-1$

Algorithm ($\mathscr{L}$ irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

1. Decide if the Riccati equation has an algebraic solution u of degree at most $(49r)^{r^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

▷ [Singer, 1979]: generalization to any input $\mathscr{L}$ $\longrightarrow$ requires LDE factoring

▷ [Singer, 2014; B., Salvy, Singer, 2024]: compute $\mathscr{L}^{\mathrm{alg}}$, factor of $\mathscr{L}$ whose solution space is spanned by alg. solutions of $\mathscr{L}$ $\longrightarrow$ requires LDE factoring

Application to Stanley's problem

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ and sufficiently many initial terms, is transcendental.

Algorithm for problem (S) [B., Salvy, Singer, 2024]

1. Compute $\mathscr{L}^{\text{alg}}$
2. Decide if $\mathscr{L}^{\text{alg}}$ annihilates f

▷ Benefit: Solves (in principle) problems **(S)**, **(F)**, **(L)**: *algebraicity is decidable*

▷ Drawbacks: Step 1 involves *impractical bounds* & *requires LDE factorization*

▷ LDE factorization is effective
[Fabry, 1885], [Markov, 1891], [Grigoriev, 1990], [van Hoeij, 1997]

▷ ...but possibly extremely costly: complexity $(N\mathcal{L})^{O(r^4)}$,
with $\mathcal{L} = \text{bitsize}(\mathscr{L})$ and $N = e^{(\mathcal{L}\cdot 2^r)^{o(2^r)}}$ [Grigoriev, 1990]

A practical method, based on Minimization

Practical method: the basic idea

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ and sufficiently many initial terms, is transcendental.

Key property: If $\mathscr{L}_f^{\min}$ has a log singularity, then f is transcendental.

▷ Pros and cons: Avoids factorization of $\mathscr{L}$, but requires to compute $\mathscr{L}_f^{\min}$.

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5$$

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

   $$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\, A_1 = 5$$

2. Conversion from recurrence to differential equation $\mathscr{L}(f) = 0$, where

   $$\mathscr{L} = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

   $$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5$$

2. Conversion from recurrence to differential equation $\mathscr{L}(f) = 0$, where

   $$\mathscr{L} = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

3. Minimization: [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2024]
   compute least-order $\mathscr{L}_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $\mathscr{L}_f^{\min}(f) = 0$

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

   $$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2+17\,n+5)A_n, \quad A_0 = 1,\ A_1 = 5$$

2. Conversion from recurrence to differential equation $\mathscr{L}(f) = 0$, where

   $$\mathscr{L} = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

3. Minimization: [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2024]
   compute least-order $\mathscr{L}_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $\mathscr{L}_f^{\min}(f) = 0$

4. Local solutions of $\mathscr{L}_f^{\min}$: [Frobenius, 1873], [Chudnovsky[2], 1987]

   $$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\ \right\}$$

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

   $$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5$$

2. Conversion from recurrence to differential equation $\mathscr{L}(f) = 0$, where

   $$\mathscr{L} = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

3. Minimization: [Adamczewski, Rivoal, 2018], [B., Rivoal, Salvy, 2024]
   compute least-order $\mathscr{L}_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $\mathscr{L}_f^{\min}(f) = 0$

4. Local solutions of $\mathscr{L}_f^{\min}$: [Frobenius, 1873], [Chudnovsky[2], 1987]

   $$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\ \right\}$$

5. Conclusion: f is transcendental[†]

---

† f algebraic would imply a full basis of algebraic solutions for $\mathscr{L}_f^{\min}$ [Tannery, 1875].

The new method: a first version

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

▷ Principle: **(S)** is reduced to **(F)** via minimization

Second algorithm for problem **(S)** [B., Salvy, Singer, 2024]

1. Compute $\mathscr{L}_f^{\min}$ [B., Rivoal, Salvy, 2024]
2. Decide if $\mathscr{L}_f^{\min}$ has only algebraic solutions; if so return A, else return T. [Singer, 1979]

▷ Benefit: Solves (in principle) Stanley's problem **(S)**: *algebraicity is decidable*

▷ Drawback: Step 2 can be very costly in practice.

The new method: a more efficient version

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

Third algorithm for problem (S) [B., Salvy, Singer, 2024]

1. Compute $\mathscr{L}_f^{\min}$ [B., Rivoal, Salvy, 2024]
2. If $\mathscr{L}_f^{\min}$ has a logarithmic singularity, return T; otherwise return A

▷ This algorithm is always correct when it returns T

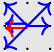
▷ *Conjecturally*, under the additional assumption that *f is globally bounded*$^{\diamondsuit}$, it is also always correct$^{\clubsuit}$ when it returns A [Christol, 1986], [André, 1997]

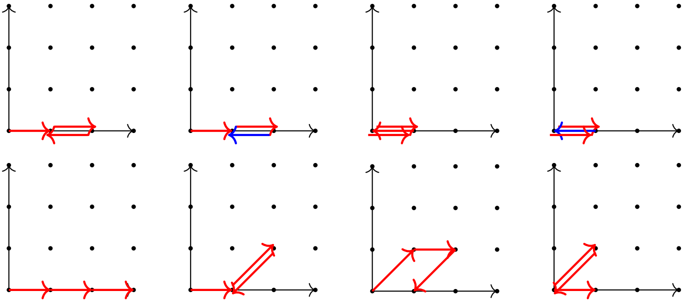
$^{\diamondsuit}$ E.g. if f is given as GF of a binomial sum, or as the diagonal of a rational function

$^{\clubsuit}$ NB: not true without the global boundedness assumption, e.g. $f(t) = {}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{5}{6} \\ \frac{7}{6} & \end{matrix} \,\middle|\, t\right)$

Ex. (C): a difficult quadrant model with repeated steps

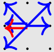
Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star, 0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation $\mathscr{L}$ for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\operatorname{ord}(\mathscr{L}_f^{\min}) \leq 10$, then $\deg_t(\mathscr{L}_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $\mathscr{L}_f^{\min} = \mathscr{L}$
5. $\mathscr{L}$ has a log singularity at $t = 0$, and so f is transcendental $\square$

Summary

- Problems **(S)**, **(F)**, **(L)** on algebraicity of solutions of LDEs are decidable
- In practice, proving *transcendence is easier* than proving algebraicity (!)
- LDE minimization is a practical alternative for proving transcendence
  - $\longrightarrow$ allows to solve difficult problems from applications
  - $\longrightarrow$ also useful in other contexts (effective Siegel-Shidlovskii)
- Guess-and-Prove is a powerful method for proving algebraicity
  - $\longrightarrow$ robust: adapts to other functional equations
  - $\longrightarrow$ main limitation: output size!
- Brute-force / naive algorithms $\longrightarrow$ hopeless on "real-life" applications

Thanks for your attention!