An overview of q-analogs of real numbers

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Based on several joint articles.
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Famous q-analogs of integers

- q-integers [Euler, Gauss, ...] $[6]_q = 1 + q + q^2 + q^3 + q^4 + q^5$

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [-n]_q = -q^{-1} - q^{-2} - \cdots - q^{-n}.$$

- q-factorials

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q \qquad [3]_q! = 1 + 2q + 2q^2 + q^3$$

- q-binomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

What can we expect from a q-analog?

- Natural definition
- q-analogs of positive integers are often polynomials
  - with positive integer coefficients
  - with symmetric sequence of coefficients (palindromes)
  - with unimodal sequence of coefficients
- Combinatorial interpretations
- Geometric interpretations

q-analogs of rationals

- q-rationals ?

Definition [MG-O, 2020]:

$$\left[\frac{r}{s}\right]_q = [c_1]_q - \cfrac{1q^{c_1-1}}{[c_2]_q - \cfrac{1q^{c_2-1}}{\ddots \, \cfrac{\ddots}{[c_{k-1}]_q - \cfrac{1q^{c_{k-1}-1}}{[c_k]_q}}}}$$

Examples:

$$\left[\frac{5}{2}\right]_q = \left[3 - \frac{1}{2}\right]_q = [3]_q - \frac{q^2}{[2]_q} = \frac{1+2q+q^2+q^3}{1+q}$$

$$\left[\frac{5}{3}\right]_q = \left[2 - \frac{1}{3}\right]_q = [2]_q - \frac{q}{[3]_q} = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

First properties

q-rationals are of the form

$$\left[\frac{r}{s}\right]_q = \pm q^N \frac{\mathcal{R}}{\mathcal{S}}$$

where $\mathcal{R}$ and $\mathcal{S}$ are coprime polynomials in q with

- positive integers coefficients
- constant and leading coefficients equal to 1
- unimodal sequence
  [McConville–Sagan–Smyth, 2021],
  [Kantarci Oguz–Ravichandran, 2021]
- NOT palindromes (but not too far ...)

First theorem

$$\left[\frac{r}{s}\right]_q = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots \cfrac{\ddots}{[c_{k-1}]_q - \cfrac{q^{c_{k-1}-1}}{[c_k]_q}}}}$$

Theorem [Leclere-MG,2021]

The rational function does not depend on the choice of the c_i's.

q-deformations of continued fractions

$$\frac{5}{3} = 2 - \frac{1}{3} \quad \to \quad 1 + q - \frac{q}{1+q+q^2}$$

$$=$$

$$= 1 - \cfrac{1}{-1 - \cfrac{1}{2}} \quad \to \quad 1 - \cfrac{1}{-q^{-1} - \cfrac{q^{-2}}{1+q}}$$

$$=$$

$$= -1 - \cfrac{1}{0 - \cfrac{1}{3 - \cfrac{1}{3}}} \quad \to \quad -q^{-1} - \cfrac{q^{-2}}{0 - \cfrac{q^{-1}}{1+q+q^2 - \cfrac{q^2}{1+q+q^2}}}$$

$$=$$

$$\frac{1+q+2q^2+q^3}{1+q+q^2} =: \left[\frac{5}{3}\right]_q$$

Regular continued fractions

In particular,

$$\left[\frac{r}{s}\right]_q = [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots \cfrac{\ddots}{[a_{2m-1}]_q + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}$$

q-reals

- q-irrationals ?

  Stabilisation Phenomenon

Stabilization Phenomenon

$$\frac{12}{5} = 2.4 \qquad \left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 3q^8 - 3q^9 - 4q^{10} + 7q^{11} + 4q^{12} \dots$$

$$\frac{70}{29} = 2.4137... \qquad \left[\frac{70}{29}\right]_q = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 6q^{12} \dots$$

$$\frac{408}{169} = 2.41420... \qquad \left[\frac{408}{169}\right]_q = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+...+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+...+4q^{10}+q^{11}}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$$

$\downarrow$

$1 + \sqrt{2}$

$=2.41421...$

Stabilization Phenomenon

$\frac{12}{5} = 2.4 \qquad \left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$

$= 1 + q + q^4 - 2q^6 + q^7 + 3q^8 - 3q^9 - 4q^{10} + 7q^{11} + 4q^{12} \dots$

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$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$

$\downarrow \qquad \downarrow$

$1+\sqrt{2}$ $=2.41421...$ $\qquad [1+\sqrt{2}]_q = 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} - 55q^{13} + 18q^{14} + 146q^{15} \dots$

$= \frac{q^3+2q-1+\sqrt{q^6+4q^4-2q^3+4q^2+1}}{2q}$

Second theorem: q-analogs of irrationals

- q-real numbers

Theorem [MG-Ovsienko, 2022]

If $(x_n)_n$ is a sequence of rational numbers converging to an irrational number x, then the sequence of the Laurent series of the q-rationals $\left([x_n]_q\right)_n$ converges to a Laurent series $[x]_q$ (that depends only on x).

Definition [MG-Ovsienko, 2022]

The q-irrationals are formal Laurent series with integer coefficients obtained from the convergence of q-rationals.

Examples of q-irrationals

$$
\begin{aligned}
[\varphi]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 \\
&\quad -82q^9 + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} \\
&\quad +5373q^{14} - 12735q^{15} + 30372q^{16} - 72832q^{17} \\
&\quad +175502q^{18} - 424748q^{19} + 1032004q^{20} \cdots
\end{aligned}
$$

$$
\begin{aligned}
[\pi]_q &= 1 + q + q^2 + q^{10} - q^{12} - q^{13} + q^{15} + q^{16} \\
&\quad -q^{20} - 2q^{21} - q^{22} + 2q^{23} + 4q^{24} + q^{25} \\
&\quad -4q^{27} - 4q^{28} - 2q^{29} + q^{30} + 5q^{31} + 8q^{32} + 3q^{33} \\
&\quad -3q^{34} - 10q^{35} - 12q^{36} - 5q^{37} + 8q^{38} + 19q^{39} + 20q^{40} \\
&\quad +2q^{41} - 18q^{42} - 32q^{43} - 25q^{44} + 31q^{46} + 51q^{47} \\
&\quad +45q^{48} - 7q^{49} - 65q^{50} - 94q^{51} - 57q^{52} + 35q^{53} \cdots
\end{aligned}
$$

$$
\begin{aligned}
[e]_q &= 1 + q + q^3 - q^5 + 2q^6 - 3q^7 + 3q^8 - q^9 \\
&\quad -3q^{10} + 9q^{11} - 17q^{12} + 25q^{13} - 29q^{14} + 23q^{15} + 2q^{16} \\
&\quad -54q^{17} + 134q^{18} - 232q^{19} + 320q^{20} - 347q^{21} + 243q^{22} + 71q^{23} \\
&\quad -660q^{24} + 1531q^{25} - 2575q^{26} + 3504q^{27} - 3804q^{28} + 2747q^{29} + 488q^{30} \cdots
\end{aligned}
$$

Radius of convergence

The radius of convergence of the series $[\varphi]_q$ is $R_\varphi := \frac{3-\sqrt{5}}{2}$.

Conjecture (Leclere–MG–Ovsienko–Veselov, 2024)

The radius of convergence R_x of the series $[x]_q$ satisfies:

$$R_x \geq R_\varphi.$$

Deformation of matrices

- q-analogs of matrices

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \to \quad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \to \quad L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$$

$$M \in \mathrm{PSL}_2(\mathbb{Z}) \quad \to \quad M_q \in \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])/\{\pm q^N I_2\}$$

This is a group isomorphism

$$\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathrm{PSL}_2(\mathbb{Z})_q$$

where $\mathrm{PSL}_2(\mathbb{Z})_q = < R_q, L_q > \quad \subset \mathrm{PGL}_2(\mathbb{Z}[q^{\pm 1}])$

$\mathrm{PSL}_2(\mathbb{Z})$-action

Theorem (Leclere-MG, 2022)

The q-deformations commute with the action of $\mathrm{PSL}_2(\mathbb{Z})$, i.e. for $M \in \mathrm{PSL}_2(\mathbb{Z})$, $x \in \mathbb{R}$

$$[M \cdot x]_q = M_q \cdot [x]_q$$

Recall:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = \frac{aX + b}{cX + d}$$

Note:
The set of rational numbers (+infinity) is the orbit of 1 under the action of $\mathrm{PSL}_2(\mathbb{Z})$

The set of q-rational numbers (+infinity) is the orbit of 1 under the action of $\mathrm{PSL}_2(\mathbb{Z})_q$

Traces of q-matrices

Theorem (Leclere-MG, 2022)

The trace of a q-matrix M_q is a polynomial with positive integer coefficients forming a palindrome.

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$$

$\mathrm{Tr} = 1+q+2q^2+q^3+q^4$

$$\begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+3q^3+3q^4+2q^5+q^6 & 1+q+2q^2+q^3 \\ q+2q^2+2q^3+q^4+q^5 & 1+q+q^2 \end{pmatrix}$$

$\mathrm{Tr} = 1+2q+3q^2+3q^3+3q^4+2q^5+q^6$

$$\begin{pmatrix} 31 & 13 \\ 19 & 8 \end{pmatrix}_q = \begin{pmatrix} q+3q^2+5q^3+6q^4+7q^5+5q^6+3q^7+q^8 & 1+2q+3q^2+3q^3+3q^4+q^5 \\ q+3q^2+4q^3+4q^4+4q^5+2q^6+q^7 & 1+2q+2q^2+2q^3+q^4 \end{pmatrix}$$

$\mathrm{Tr} = 1+3q+5q^2+7q^3+7q^4+7q^5+5q^6+3q^7+q^8$

$\rightarrow$ q-Markov [Kogiso, 2020], [Labbé–Lapointe, 2021], [L–L–Steiner, 2023]

Million of questions

- Combinatorial interpretations ?
- Geometric interpretations ?
- Relations with quantum invariants ?
- Quantum calculus ?
- Properties of the power series ?
- q-complex numbers ?
- q-matrices in $\mathrm{SL}_n(\mathbb{Z})$? $\mathrm{GL}_n(\mathbb{Z})$?
- ...

Twin brother: “left” deformation

The stabilization phenomenon fails for a rational number x !

If $x_n > x$ then $\lim\, [x_n]_q = [x]_q$,

if $x_n < x$ then $\lim\, [x_n]_q = [x]^\flat_q \neq [x]_q$.

Example:

$$\lim\ \left[1 + \frac{1}{n}\right]_q = 1, \qquad \lim\ \left[1 - \frac{1}{n}\right]_q = q.$$

[Bapat–Becker–Licata, 2022] gave explicitit formulas for $[x]^\flat_q$,

In particular for $n \in \mathbb{N}$ one has

$$[n]^\flat_q = 1 + q + \ldots + q^{n-2} + q^n.$$

and e.g $\left[\frac{5}{2}\right]^\flat_q = \left[3 - \frac{1}{2}\right]_q = [3]_q - \frac{q^2}{[2]^\flat_q} = \frac{1 + q + q^2 + q^3 + q^4}{1 + q^2}$

Twin brother: “left” deformation

[BBL, 2022]: The set of “left” q-rational numbers (+infinity) is the orbit of q under the action of $\mathrm{PSL}_2(\mathbb{Z})_q$.

Theorem (Jouteur 2025)

The q-deformations $[\ ]_q$ and $[\ ]_q^\flat$ are the only two deformations that commute with the action of $\mathrm{PSL}_2(\mathbb{Z})$.

Twin brother: "left" deformation

Theorem (Jouteur 2025)

For $x \in \mathbb{Q}$ *one has*

$$\left[\frac{1}{x}\right]_q = \frac{(q-1)[x]^\flat_q + 1}{q[x]^\flat_q + 1 - q},$$

and for $x \in \mathbb{R} \setminus \mathbb{Q}$ *one has*

$$\left[\frac{1}{x}\right]_q = \frac{(q-1)[x]_q + 1}{q[x]_q + 1 - q},$$

New symmetry:

$$J_q = \begin{pmatrix} q-1 & 1 \\ q & 1-q \end{pmatrix}$$

leading to a deformation $\mathrm{GL}_2(\mathbb{Z}) \simeq < R_q, L_q, J_q >$.

References

with Valentin Ovsienko:

- q-deformed rationals and q-continued fractions. Forum Math. Sigma (2020)
- On q-deformed real numbers. Exp. Math. (2022)

with Ludivine Leclere:

- q-deformations in the modular group and of the real quadratic irrational numbers. Adv. Appl. Math. (2021).
- Quantum continuants, quantum rotundus and triangulations of annuli. Electron. J. Combin. (2023)

with LL-VO-Alexander Veselov:

- On radius of convergence of q-deformed real numbers.Mosc. Math. J. (2024).

with VO-AV:

- Burau representation of braid groups and q-rationals IMRN (2024).

Related work...

- Knots invariants
  [Lee–Schiffler, 2019] [Kogiso–Wakui, 2019] [Sikora, 2020]
- Enumerative interpretations
  [McConville–Sagan–Smyth, 2021] [Ovenhouse, 2021]
  [Kantarci Oguz–Ravichandran, 2021] [Kantarci Oguz–Yildirim, 2022]
- *q*-Markov
  [Kogiso, 2020] [Labbé–Lapointe, 2021] [Labbé–Lapointe–Steiner, 2023]
- Categorical and homological interpretations
  [Bapat–Becker–Licata, 2022] [L.Fan–Y.Qiu, 2023]
- Higher versions
  [Burcroff, Ovenhouse, Schiffler, Zhang, 2024] [Jouteur, 2024]
- *q*-series, *q*-calculus, quantum algebra,...
  [Machacek–Ovenhouse, 2021] [Ovsienko, 2021] [A. Thomas, 2023]
  [Ovsienko-Pedon, 2023] [Ovsienko-Ustinov, 2024] [Jouteur 2025]
  [Lasker 2024] [Evans–Veselov–Winn, 2024]

Thanq You !