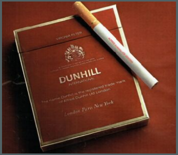
Computer algebra tools for solving combinatorial functional equations

Mohab Safey El Din

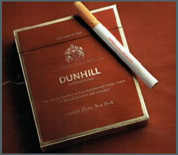
SORBONNE UNIVERSITÉ
cnrs
LIP6

Joint works with
Alin Bostan and Hadrien Notarantonio

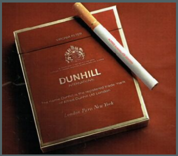
Combinatorics and functional equations



Combinatorics and functional equations



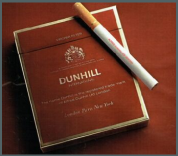
Combinatorics and functional equations



$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

Combinatorics and functional equations



$a_{n,d}$ = number of paths leading to position d after n steps

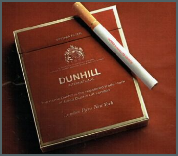
$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\, G_u + t\left(\frac{G_u - G_0}{u}\right)}$$

Fixed-point type equation

Discrete differential operator

Combinatorics and functional equations



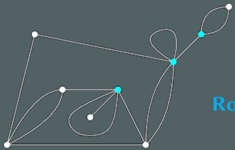
$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\, G_u + t\left(\frac{G_u - G_0}{u}\right)}$$

Discrete differential operator

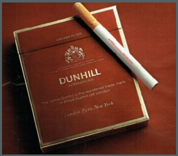
Fixed-point type equation



Rooted planar maps

$$G_u = 1 + ut\left(uG_u^2 + \frac{uG_u - G_1}{u-1}\right)$$

Combinatorics and functional equations



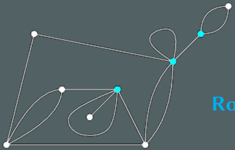
$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\, G_u + t\left(\frac{G_u - G_0}{u}\right)}$$

Fixed-point type equation

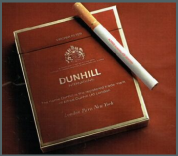
Discrete differential operator



Rooted planar maps

$$G_u = 1 + ut\left(uG_u^2 + \frac{uG_u - G_1}{u-1}\right) = 1 + ut\left(uG_u^2 + u\frac{G_u - G_1}{u-1} + G_1\right)$$

Combinatorics and functional equations



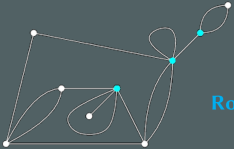
$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\, G_u + t \left(\frac{G_u - G_0}{u}\right)}$$

Fixed-point type equation

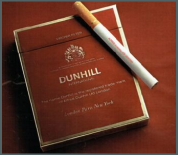
Discrete differential operator



Rooted planar maps

$$G_u = 1 + u^2 t\, G_u^2 + u^2 t \underbrace{\frac{G_u - G_1}{u - 1}}_{=:\mathrm{d}_1^{(1)} G_u} + u\, G_1$$

Combinatorics and functional equations



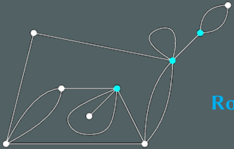
$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\,G_u + t\,\left(\frac{G_u - G_0}{u}\right)}$$

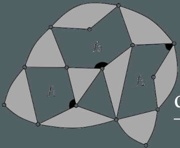
Fixed-point type equation

Discrete differential operator



Rooted planar maps

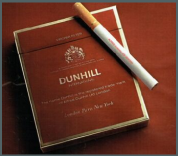
$$G_u = 1 + u^2 t\, G_u^2 + u^2 t \underbrace{\frac{G_u - G_1}{u-1}}_{=:\mathrm{d}_1^{(1)} G_u} + u\, G_1$$



3-constellations

$$G_u = 1 + utG_u^3 + (2G + G_1)\mathrm{d}_1^{(1)} G_u + \underbrace{\frac{\mathrm{d}_1^{(1)} G_u - (u-1)\,(\partial_u G_u)_1}{(u-1)^2}}_{\mathrm{d}_1^{(2)} G_u}$$

Combinatorics and functional equations



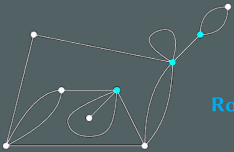
$a_{n,d}$ = number of paths leading to position d after n steps

$$G_{u,t} = \sum_{n,d\geq 0} a_{n,d} u^d t^n \text{ with } a_{n,d} = 0 \text{ when } d > n \rightsquigarrow G \in \mathbb{Q}[u][[t]]$$

$$a_{n,d} = a_{n-1,d-1} + a_{n-1,d+1} \rightsquigarrow \boxed{G_u = 1 + ut\, G_u + t\left(\frac{G_u - G_0}{u}\right)}$$

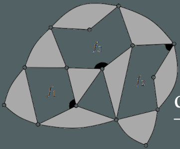
Fixed-point type equation

Discrete differential operator



Rooted planar maps

$$G_u = 1 + u^2 t\, G_u^2 + u^2 t \underbrace{\frac{G_u - G_1}{u-1}}_{=:\mathrm{d}_1^{(1)} G_u} + u\, G_1$$



3-constellations

$$G_u = 1 + utG_u^3 + (2G + G_1)\mathrm{d}_1^{(1)} G_u + \underbrace{\frac{\mathrm{d}_1^{(1)} G_u - (u-1)\left(\partial_u G_u\right)_1}{(u-1)^2}}_{\mathrm{d}_1^{(2)} G_u}$$

Triangulations **Lattices**

Albenque/Ménard/Schaeffer, Banderier/Flajolet, Bousquet-Mélou/Jéhanne, Temperley, Tutte, Zeilberger, [...]

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)}G_u, \ldots, \mathrm{d}_a^{(k)}G_u, t, u\right)$$ with $\mathscr{U} \in \mathbb{Q}[u]$ and $\mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)} G_u, \ldots, \mathrm{d}_a^{(k)} G_u, t, u\right) \text{ with } \mathscr{U} \in \mathbb{Q}[u] \text{ and}$$
$$\mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$$

Properties of G_u? $\rightsquigarrow$ asymptotics of G_a

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)}G_u, \ldots, \mathrm{d}_a^{(k)}G_u, t, u\right)$$ with $\mathscr{U} \in \mathbb{Q}[u]$ and $\mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$

Properties of G_u? $\rightsquigarrow$ asymptotics of G_a

Algebraicity result (Bousquet-Mélou/Jéhanne)
There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)} G_u, \ldots, \mathrm{d}_a^{(k)} G_u, t, u\right)$$ with $\mathscr{U} \in \mathbb{Q}[u]$ and $\mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$

Properties of G_u? $\rightsquigarrow$ asymptotics of G_a

Compute $\mathscr{A}$?

Quantitative aspects?

Algebraicity result (**Bousquet-Mélou/Jéhanne**)
There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)} G_u, \ldots, \mathrm{d}_a^{(k)} G_u, t, u\right) \text{ with } \mathscr{U} \in \mathbb{Q}[u] \text{ and } \mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$$

Properties of G_u? $\rightsquigarrow$ asymptotics of G_a

Compute $\mathscr{A}$?

Quantitative aspects?

Algebraicity result (Bousquet-Mélou/Jéhanne)
There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

$$G_u = \sum_{i \geq 1} (u-a)^{i-1} g_i \rightsquigarrow \mathrm{d}_a^{(k)} G_u = \frac{G_u - g_1 - (u-a) g_2 - \cdots - (u-a)^{k-1} g_k}{u-a}$$

$$\mathscr{P}\underbrace{(G_u, g_1, \ldots, g_k, t, u)}_{S} \equiv 0 \quad \text{with } \mathscr{P} \in \mathbb{Q}[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, t, u]$$

Functional equations and polynomials

$$G_u = \mathscr{U} + t\mathscr{Q}\left(G_u, \mathrm{d}_a^{(1)}G_u, \ldots, \mathrm{d}_a^{(k)}G_u, t, u\right)$$ with $\mathscr{U} \in \mathbb{Q}[u]$ and $\mathscr{Q} \in \mathbb{Q}[\gamma, \delta_1, \ldots, \delta_k, t, u]$

Properties of G_u? $\rightsquigarrow$ asymptotics of G_a

Compute $\mathscr{A}$?

Quantitative aspects?

Algebraicity result (**Bousquet-Mélou/Jéhanne**)
There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

$$G_u = \sum_{i\geq 1}(u-a)^{i-1}g_i \rightsquigarrow \mathrm{d}_a^{(k)}G_u = \frac{G_u - g_1 - (u-a)g_2 - \cdots - (u-a)^{k-1}g_k}{u-a}$$

$$\mathscr{P}\underbrace{\left(G_u, g_1, \ldots, g_k, t, u\right)}_{S} \equiv 0$$ with $\mathscr{P} \in \mathbb{Q}[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, t, u]$

☛ differentiate w.r.t. $u \rightsquigarrow \frac{\partial G_u}{\partial u}\frac{\partial \mathscr{P}}{\partial \kappa}(S) + \frac{\partial \mathscr{P}}{\partial u}(S) = 0$

Functional equations and polynomial systems (II)

Bousquet-Mélou/Jéhanne

We have $\mathscr{P}\underbrace{(G_u, g_1, \ldots, g_k, t, u)}_{S} \equiv 0$ with $\mathscr{P} \in \mathbb{Q}[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, t, u]$

☛ differentiate w.r.t. $u \rightsquigarrow \frac{\partial G_u}{\partial u} \frac{\partial \mathscr{P}}{\partial \kappa}(S) + \frac{\partial \mathscr{P}}{\partial u}(S) = 0$

Functional equations and polynomial systems (II)

Bousquet-Mélou/Jéhanne

We have $\mathscr{P}\underbrace{(G_u, g_1, \ldots, g_k, t, u)}_{S} \equiv 0$ with $\mathscr{P} \in \mathbb{Q}[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, t, u]$

☛ differentiate w.r.t. $u \rightsquigarrow \frac{\partial G_u}{\partial u} \frac{\partial \mathscr{P}}{\partial \kappa}(S) + \frac{\partial \mathscr{P}}{\partial u}(S) = 0$

If $\exists U_1, \ldots, U_k$ distinct fractional power series in t such that $\frac{\partial \mathscr{P}}{\partial \kappa}(S(U_i)) = 0$ then

$$\begin{aligned}
\mathscr{P}(\kappa_1, \underline{\gamma}, t, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, t, u_2) &= 0 & & & \mathscr{P}(\kappa_k, \underline{\gamma}, t, u_k) &= 0\\
\frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, t, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, t, u_2) &= 0 & & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, t, u_k) &= 0\\
\frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, t, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, t, u_2) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, t, u_k) &= 0
\end{aligned}$$

these $3k$ equations involving $k + 2k + 1$ indeterminates fully determine the U_i's (under some transversality conditions)

Functional equations and polynomial systems (II)

Bousquet-Mélou/Jéhanne

We have $\mathscr{P}\underbrace{(G_u, g_1, \ldots, g_k, t, u)}_{S} \equiv 0$ with $\mathscr{P} \in \mathbb{Q}[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, t, u]$

☛ differentiate w.r.t. $u \rightsquigarrow \frac{\partial G_u}{\partial u}\frac{\partial \mathscr{P}}{\partial \kappa}(S) + \frac{\partial \mathscr{P}}{\partial u}(S) = 0$

If $\exists U_1, \ldots, U_k$ distinct fractional power series in t such that $\frac{\partial \mathscr{P}}{\partial \kappa}(S(U_i)) = 0$ then

$$\begin{array}{lllll}
\mathscr{P}(\kappa_1, \underline{\gamma}, t, u_1) = 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, t, u_2) = 0 & & \mathscr{P}(\kappa_k, \underline{\gamma}, t, u_k) = 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, t, u_1) = 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, t, u_2) = 0 & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, t, u_k) = 0 \\
\frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, t, u_1) = 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, t, u_2) = 0 & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, t, u_k) = 0
\end{array}$$

these $3k$ equations involving $k + 2k + 1$ indeterminates fully determine the U_i's (under some transversality conditions)

➡ **All unknown series are algebraic**

✓**These distinct series do exist** Bousquet-Mélou/Jéhanne

✓fixed-point type equation $\Longrightarrow \mathscr{A}$ such that $\mathscr{A}(G_a, t) = \mathscr{A}(\gamma_1 = G_a, t) \equiv 0$

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

$$\mathscr{P} \in \mathbb{Q}(t)[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, u] \rightsquigarrow \dim(\text{Zeroes}(\mathscr{P}, \mathbb{K})) = k + 1 \qquad \mathbb{K} = \overline{\mathbb{Q}(t)}$$

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

$$\mathscr{P} \in \mathbb{Q}(t)[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, u] \rightsquigarrow \dim\left(\text{Zeroes}(\mathscr{P}, \mathbb{K})\right) = k + 1 \qquad \mathbb{K} = \overline{\mathbb{Q}(t)}$$

$$\dim\left(\text{Zeroes}\left(\mathscr{P}, \frac{\partial \mathscr{P}}{\partial \kappa}, \frac{\partial \mathscr{P}}{\partial u}\right)\right) = k - 1 \qquad \text{expectedly}$$

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

$\mathscr{P} \in \mathbb{Q}(t)[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, u] \rightsquigarrow \dim(\text{Zeroes}(\mathscr{P}, \mathbb{K})) = k + 1$ $\qquad \mathbb{K} = \overline{\mathbb{Q}(t)}$

$\dim\left(\text{Zeroes}\left(\mathscr{P}, \frac{\partial \mathscr{P}}{\partial \kappa}, \frac{\partial \mathscr{P}}{\partial u}\right)\right) = k - 1$ expectedly

? For which set of values $\mathbf{g}$ of $\underline{\gamma}$, are there k distinct u-coordinate solutions to $\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0$?

Geometry of the problem (I)

Dimension number of degrees of freedom one can move on the solution set

$\mathscr{P} \in \mathbb{Q}(t)[\kappa, \underbrace{\gamma_1, \ldots, \gamma_k}_{\underline{\gamma}}, u] \rightsquigarrow \dim(\text{Zeroes}(\mathscr{P}, \mathbb{K})) = k+1$ $\quad \mathbb{K} = \overline{\mathbb{Q}(t)}$

$\dim\left(\text{Zeroes}\left(\mathscr{P}, \frac{\partial \mathscr{P}}{\partial \kappa}, \frac{\partial \mathscr{P}}{\partial u}\right)\right) = k-1$ expectedly

? For which set of values $\mathbf{g}$ of $\underline{\gamma}$, are there k distinct u-coordinate solutions to
$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0$?

Duplicate variables

Geometric methods

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to

$$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0?$$

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to
$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0$?

Example 1. Take 3 linear equations in $\mathbb{Q}[\kappa, \gamma_1, u]$
$\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{Q}_3 = 0$
generically **no solution**

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to
$$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0?$$

Example 1. Take 3 linear equations in $\mathbb{Q}[\kappa, \gamma_1, u]$
$\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{Q}_3 = 0$
generically **no solution**

determinantal formulation
Univariate polynomial in γ_1

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to
$$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0?$$

Example 1. Take 3 linear equations in $\mathbb{Q}[\kappa, \gamma_1, u]$
$\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{Q}_3 = 0$
generically **no solution**

determinantal formulation
Univariate polynomial in γ_1

Example 2. $\kappa = 1$
$${\gamma_1}^2 - {\gamma_2}^2 = \gamma_1 u - \gamma_2 = u^2 - 1 = 0$$
Solution set is a **curve**

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to
$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0$?

Example 1. Take 3 linear equations in $\mathbb{Q}[\kappa, \gamma_1, u]$
$\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{Q}_3 = 0$
generically **no solution**

determinantal formulation
Univariate polynomial in γ_1

Example 2. $\kappa = 1$
$\gamma_1{}^2 - \gamma_2{}^2 = \gamma_1 u - \gamma_2 = u^2 - 1 = 0$
Solution set is a **curve**

$\gamma_1 = \gamma_2 = 0$

Geometry of the problem (II)

? For which set of values $\mathbf{g} = (g_1, \ldots, g_k)$ of $\underline{\gamma}$,
are there k distinct u-coordinate solutions to
$\mathscr{P}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa, \mathbf{g}, u) = \frac{\partial \mathscr{P}}{\partial u}(\kappa, \mathbf{g}, u) = 0$?

Example 1. Take 3 linear equations in $\mathbb{Q}[\kappa, \gamma_1, u]$
$\mathscr{Q}_1 = \mathscr{Q}_2 = \mathscr{Q}_3 = 0$
generically **no solution**

determinantal formulation
Univariate polynomial in γ_1

Example 2. $\kappa = 1$
$\gamma_1{}^2 - \gamma_2{}^2 = \gamma_1 u - \gamma_2 = u^2 - 1 = 0$
Solution set is a **curve**

$\gamma_1 = \gamma_2 = 0$

These subsets are **exceptional**

Direct solving – Gröbner bases

$$\begin{aligned}
\mathscr{P}\left(\kappa_1, \underline{\gamma}, t, u_1\right) &= 0 & \mathscr{P}\left(\kappa_2, \underline{\gamma}, t, u_2\right) &= 0 & & & \mathscr{P}\left(\kappa_k, \underline{\gamma}, t, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_1, \underline{\gamma}, t, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_2, \underline{\gamma}, t, u_2\right) &= 0 & \cdots\cdots & & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_k, \underline{\gamma}, t, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial u}\left(\kappa_1, \underline{\gamma}, t, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_2, \underline{\gamma}, t, u_2\right) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_k, \underline{\gamma}, t, u_k\right) &= 0
\end{aligned}$$

Direct solving – Gröbner bases

$$\begin{aligned}
\mathscr{P}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \mathscr{P}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & & \mathscr{P}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & \cdots\cdots & & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial u}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0
\end{aligned}$$

Direct solving – Gröbner bases

$$
\begin{aligned}
\mathscr{P}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \mathscr{P}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & \mathscr{P}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & \cdots\cdots \quad & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial u}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0
\end{aligned}
$$

Completion mechanism to discover hidden relations

Direct solving – Gröbner bases

$$\begin{aligned}
\mathscr{P}\left(\kappa_1,\underline{\gamma},u_1\right) &= 0 & \mathscr{P}\left(\kappa_2,\underline{\gamma},u_2\right) &= 0 & & \mathscr{P}\left(\kappa_k,\underline{\gamma},u_k\right) &= 0\\
\frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_1,\underline{\gamma},u_1\right) &= 0 & \frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_2,\underline{\gamma},u_2\right) &= 0 & \cdots\cdots & \frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_k,\underline{\gamma},u_k\right) &= 0\\
\frac{\partial\mathscr{P}}{\partial u}\left(\kappa_1,\underline{\gamma},u_1\right) &= 0 & \frac{\partial\mathscr{P}}{\partial u}\left(\kappa_2,\underline{\gamma},u_2\right) &= 0 & & \frac{\partial\mathscr{P}}{\partial u}\left(\kappa_k,\underline{\gamma},u_k\right) &= 0
\end{aligned}$$

Completion mechanism to discover hidden relations

$$\kappa = 1$$
$${\gamma_1}^2 - {\gamma_2}^2 = 0$$
$$\gamma_1 u - \gamma_2 = 0$$
$$u^2 - 1 = 0$$

Direct solving – Gröbner bases

$$
\begin{aligned}
\mathscr{P}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \mathscr{P}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & \mathscr{P}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0 \\
\frac{\partial \mathscr{P}}{\partial u}\left(\kappa_1, \underline{\gamma}, u_1\right) &= 0 & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_2, \underline{\gamma}, u_2\right) &= 0 & & \frac{\partial \mathscr{P}}{\partial u}\left(\kappa_k, \underline{\gamma}, u_k\right) &= 0
\end{aligned}
$$

Completion mechanism to discover hidden relations

$$
\begin{aligned}
&\kappa_1 = 1 \\
&{\gamma_1}^2 - {\gamma_2}^2 = 0 \\
&\gamma_1 u_1 - \gamma_2 = 0 \\
&{u_1}^2 - 1 = 0
\end{aligned}
$$

Direct solving – Gröbner bases

$$
\begin{aligned}
\mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0\\
\frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0\\
\frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0
\end{aligned}
$$

Completion mechanism to discover hidden relations

$$
\begin{aligned}
\kappa_1 &= 1 & \kappa_2 &= 1\\
{\gamma_1}^2 - {\gamma_2}^2 &= 0 & {\gamma_1}^2 - {\gamma_2}^2 &= 0\\
\gamma_1 u_1 - \gamma_2 &= 0 & \gamma_1 u_2 - \gamma_2 &= 0\\
{u_1}^2 - 1 &= 0 & {u_2}^2 - 1 &= 0
\end{aligned}
$$

Direct solving – Gröbner bases

$$\mathscr{P}\left(\kappa_1,\underline{\gamma},u_1\right)=0 \quad \mathscr{P}\left(\kappa_2,\underline{\gamma},u_2\right)=0 \quad \mathscr{P}\left(\kappa_k,\underline{\gamma},u_k\right)=0$$

$$\frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_1,\underline{\gamma},u_1\right)=0 \quad \frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_2,\underline{\gamma},u_2\right)=0 \quad \cdots\cdots \quad \frac{\partial\mathscr{P}}{\partial\kappa}\left(\kappa_k,\underline{\gamma},u_k\right)=0$$

$$\frac{\partial\mathscr{P}}{\partial u}\left(\kappa_1,\underline{\gamma},u_1\right)=0 \quad \frac{\partial\mathscr{P}}{\partial u}\left(\kappa_2,\underline{\gamma},u_2\right)=0 \quad \frac{\partial\mathscr{P}}{\partial u}\left(\kappa_k,\underline{\gamma},u_k\right)=0$$

Completion mechanism to discover hidden relations

$$\kappa_1=1 \qquad \kappa_2=1$$

$$\gamma_1{}^2-\gamma_2{}^2=0 \qquad \gamma_1{}^2-\gamma_2{}^2=0$$

$$\boxed{\gamma_1u_1-\gamma_2=0} \qquad \gamma_1u_2-\gamma_2=0$$

$$u_1{}^2-1=0 \qquad u_2{}^2-1=0$$

Direct solving – Gröbner bases

$$\begin{aligned}\mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0\\ \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0\\ \frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0\end{aligned}$$

Completion mechanism to discover hidden relations

$$\begin{aligned}\kappa_1 &= 1 & \kappa_2 &= 1\\ {\gamma_1}^2 - {\gamma_2}^2 &= 0 & {\gamma_1}^2 - {\gamma_2}^2 &= 0\\ \boxed{\gamma_1 u_1 - \gamma_2 = 0} & & \gamma_1 u_2 - \gamma_2 &= 0\\ {u_1}^2 - 1 &= 0 & {u_2}^2 - 1 &= 0\end{aligned}$$

$$\gamma_1 u_1 - \gamma_2 = 0 \Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0$$

Direct solving – Gröbner bases

$$\begin{aligned}
\mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & \cdots\cdots & & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\
\frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0
\end{aligned}$$

Completion mechanism to discover hidden relations

$$\begin{aligned}
\kappa_1 &= 1 & \kappa_2 &= 1 \\
{\gamma_1}^2 - {\gamma_2}^2 &= 0 & {\gamma_1}^2 - {\gamma_2}^2 &= 0 \\
\boxed{\gamma_1 u_1 - \gamma_2 = 0} & & \boxed{\gamma_1 u_2 - \gamma_2 = 0} & \\
{u_1}^2 - 1 &= 0 & {u_2}^2 - 1 &= 0
\end{aligned}$$

$$\gamma_1 u_1 - \gamma_2 = 0 \Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0$$

Direct solving – Gröbner bases

$$\begin{aligned} \mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & \cdots\cdots & & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0 \end{aligned}$$

Completion mechanism to discover hidden relations

$$\begin{aligned} &\kappa_1 = 1 & &\kappa_2 = 1 \\ &{\gamma_1}^2 - {\gamma_2}^2 = 0 & &{\gamma_1}^2 - {\gamma_2}^2 = 0 \\ &\boxed{\gamma_1 u_1 - \gamma_2 = 0} & &\boxed{\gamma_1 u_2 - \gamma_2 = 0} \\ &{u_1}^2 - 1 = 0 & &{u_2}^2 - 1 = 0 \end{aligned}$$

$$\gamma_1 u_1 - \gamma_2 = 0 \Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0$$

$$\gamma_1 u_2 - \gamma_2 = 0 \Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_1 = 0$$

Direct solving – Gröbner bases

$$\begin{aligned} \mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0 \end{aligned}$$

Completion mechanism to discover hidden relations

$$\begin{aligned} \kappa_1 &= 1 & \kappa_2 &= 1 \\ {\gamma_1}^2 - {\gamma_2}^2 &= 0 & {\gamma_1}^2 - {\gamma_2}^2 &= 0 \\ \boxed{\gamma_1 u_1 - \gamma_2 = 0} & & \boxed{\gamma_1 u_2 - \gamma_2 = 0} & \\ {u_1}^2 - 1 &= 0 & {u_2}^2 - 1 &= 0 \end{aligned}$$

$$\begin{aligned} \gamma_1 u_1 - \gamma_2 = 0 &\implies \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0 \\ \gamma_1 u_2 - \gamma_2 = 0 &\implies \gamma_1 u_1 u_2 - \gamma_2 u_1 = 0 \\ &\implies \gamma_2 u_2 - \gamma_2 u_1 = 0 \end{aligned}$$

Direct solving – Gröbner bases

$$\begin{aligned} \mathscr{P}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) &= 0 & \cdots\cdots & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) &= 0 \\ \frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) &= 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) &= 0 & & \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) &= 0 \end{aligned}$$

Completion mechanism to discover hidden relations

$$\begin{aligned} \kappa_1 &= 1 & \kappa_2 &= 1 \\ {\gamma_1}^2 - {\gamma_2}^2 &= 0 & {\gamma_1}^2 - {\gamma_2}^2 &= 0 \\ \boxed{\gamma_1 u_1 - \gamma_2 = 0} & & \boxed{\gamma_1 u_2 - \gamma_2 = 0} & \\ {u_1}^2 - 1 &= 0 & {u_2}^2 - 1 &= 0 \end{aligned}$$

$$\begin{aligned} \gamma_1 u_1 - \gamma_2 = 0 &\Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0 \\ \gamma_1 u_2 - \gamma_2 = 0 &\Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_1 = 0 \\ &\Longrightarrow \gamma_2 u_2 - \gamma_2 u_1 = 0 \\ &\Longrightarrow \gamma_2 = \gamma_1 = 0 \end{aligned}$$

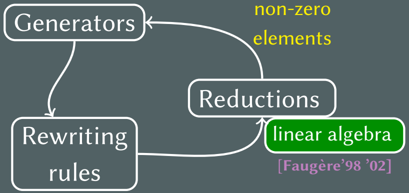
Direct solving – Gröbner bases

$$
\begin{array}{lll}
\mathscr{P}(\kappa_1, \underline{\gamma}, u_1) = 0 & \mathscr{P}(\kappa_2, \underline{\gamma}, u_2) = 0 & \qquad \mathscr{P}(\kappa_k, \underline{\gamma}, u_k) = 0 \\
\frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_1, \underline{\gamma}, u_1) = 0 & \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_2, \underline{\gamma}, u_2) = 0 & \cdots\cdots \frac{\partial \mathscr{P}}{\partial \kappa}(\kappa_k, \underline{\gamma}, u_k) = 0 \\
\frac{\partial \mathscr{P}}{\partial u}(\kappa_1, \underline{\gamma}, u_1) = 0 & \frac{\partial \mathscr{P}}{\partial u}(\kappa_2, \underline{\gamma}, u_2) = 0 & \qquad \frac{\partial \mathscr{P}}{\partial u}(\kappa_k, \underline{\gamma}, u_k) = 0
\end{array}
$$

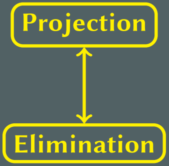
Completion mechanism to discover hidden relations

$$
\begin{array}{ll}
\kappa_1 = 1 & \kappa_2 = 1 \\
{\gamma_1}^2 - {\gamma_2}^2 = 0 & {\gamma_1}^2 - {\gamma_2}^2 = 0 \\
\boxed{\gamma_1 u_1 - \gamma_2 = 0} & \boxed{\gamma_1 u_2 - \gamma_2 = 0} \\
{u_1}^2 - 1 = 0 & {u_2}^2 - 1 = 0
\end{array}
$$

$$
\begin{aligned}
\gamma_1 u_1 - \gamma_2 = 0 &\Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_2 = 0 \\
\gamma_1 u_2 - \gamma_2 = 0 &\Longrightarrow \gamma_1 u_1 u_2 - \gamma_2 u_1 = 0 \\
&\Longrightarrow \gamma_2 u_2 - \gamma_2 u_1 = 0 \\
&\Longrightarrow \gamma_2 = \gamma_1 = 0
\end{aligned}
$$



$$
\begin{array}{l}
\mathscr{A}(\gamma_1) \\
\underline{\mathscr{A}_2}(\gamma_1, \gamma_2) \\
\vdots \\
\underline{\mathscr{A}_k}(\gamma_1, \ldots, \gamma_k) \\
\underline{\mathscr{A}_{u_1}}(\gamma_1, \ldots, \gamma_k, u_1) \\
\vdots
\end{array}
$$



Computing Gröbner bases

Computing Gröbner bases

Computing Gröbner bases

Computing Gröbner bases

Gaussian elimination

Computing Gröbner bases

f —preconditioning / degree order→ $GB_{grevlex}$ —change of order / lexicographic→ GB_{LEX}

Gaussian elimination

Bases of
$(q_1, \ldots, q_s) \to \sum_{i=1}^{s} q_i f_i$
$\deg(q_i f_i) \leq B$
Macaulay map

Computing Gröbner bases

f —— preconditioning / degree order ——→ $GB_{grevlex}$ —— change of order / lexicographic ——→ GB_{LEX}

Gaussian elimination

Bases of
$(q_1, \ldots, q_s) \to \sum_{i=1}^{s} q_i f_i$
$\deg(q_i f_i) \ \leq \ B$
Macaulay map

Characteristic polynomial

- Algorithms and termination criteria
- Generating (Hilbert) series of Macaulay maps + sparsity
- Complexity in the **generic** case

Computing Gröbner bases

f —preconditioning / degree order→ $GB_{grevlex}$ —change of order / lexicographic→ GB_{LEX}

Gaussian elimination

Bases of
$(q_1, \dots, q_s) \to \sum_{i=1}^{s} q_i f_i$
$\deg(q_i f_i) \leq B$
Macaulay map

Characteristic polynomial

- Algorithms and termination criteria
- Generating (Hilbert) series of Macaulay maps + sparsity
- Complexity in the **generic** case

Generic case $\rightsquigarrow O\left(\binom{n+\mathbb{D}_{\mathrm{reg}}}{n}^{\omega} + (\sharp\mathrm{sols})^{\omega}\right)$

Quantitative aspects

n variables, degree d

$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$

$1 \le i \le n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$

$1 \le j_k \le d$

$\boxed{d^n}$ solutions

Quantitative aspects

n variables, degree d

$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$ $\quad$ $\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$ $\quad$ $\boxed{d^n}$ solutions

$1 \leq i \leq n$ $\quad$ $1 \leq j_k \leq d$

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes

Quantitative aspects

n variables, degree d

$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$

$1 \leq i \leq n$

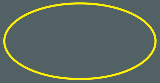
$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$

$1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Quantitative aspects

n variables, degree d

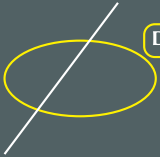
$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$
$1 \leq i \leq n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$
$1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 2

Quantitative aspects

n variables, degree d

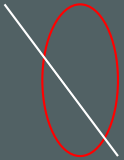
$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$
$1 \leq i \leq n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$
$1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 2

Quantitative aspects

n variables, degree d

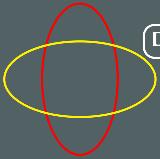
$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$
$1 \leq i \leq n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$
$1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 4

Quantitative aspects

n variables, degree d

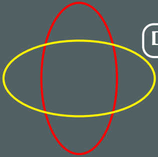
$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$ $1 \leq i \leq n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$ $1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 4

Bézout theorem

$$\deg(V \cap W) \leq \deg(V) \times \deg(W)$$

Quantitative aspects

n variables, degree d

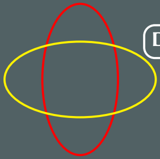
$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$, $1 \leq i \leq n$

$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$, $1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 4

Bézout theorem

$$\deg(V \cap W) \leq \deg(V) \times \deg(W)$$

Duplication

$Z = \text{Zeroes}\left(\mathscr{P}, \frac{\partial \mathscr{P}}{\partial \kappa}, \frac{\partial \mathscr{P}}{\partial u}\right)$

$$\deg(\underbrace{Z \times \cdots \times Z}_{k \text{ times}}) \leq \deg(Z)^k$$

Quantitative aspects

n variables, degree d

$f_i = \ell_{i,1} \times \cdots \times \ell_{i,d}$

$1 \leq i \leq n$

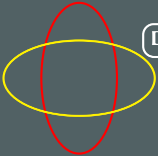
$\{\ell_{1,j_1} = \cdots = \ell_{n,j_n} = 0\}$

$1 \leq j_k \leq d$

$\boxed{d^n}$ solutions

Degree Take $V \subset \mathbb{C}^n$ an algebraic set of dimension m.

number of intersection points of V with m generic hyperplanes



Degree 4

Bézout theorem

$$\deg(V \cap W) \leq \deg(V) \times \deg(W)$$

Duplication

$Z = \text{Zeroes}\left(\mathscr{P}, \frac{\partial \mathscr{P}}{\partial \kappa}, \frac{\partial \mathscr{P}}{\partial u}\right)$

$\deg(\underbrace{Z \times \cdots \times Z}_{k \text{ times}}) \leq \deg(Z)^k$

plain C library

Berthomieu, Eder, Neiger, S.

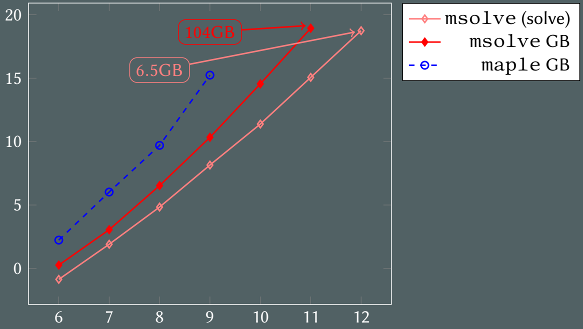
$\simeq$ 55 000 lines, license GPLv2+

uses GMP and FLINT

https://msolve.lip6.fr

https://github.com/algebraic-solving/msolve

Quantitative aspects



plain C library Berthomieu, Eder, Neiger, S.

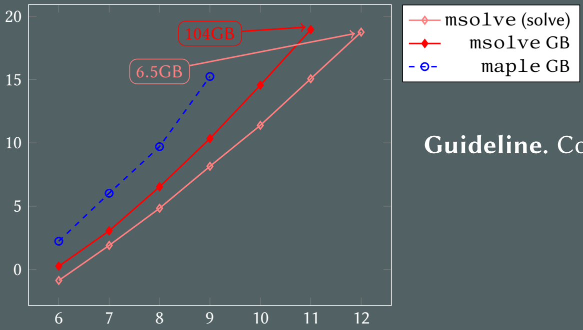
$\simeq$ 55 000 lines, license GPLv2+

uses GMP and FLINT

https://msolve.lip6.fr

https://github.com/algebraic-solving/msolve

Quantitative aspects



Guideline. Compute only what you need(!)

plain C library — Berthomieu, Eder, Neiger, S.

$\simeq$ 55 000 lines, license GPLv2+

uses GMP and FLINT

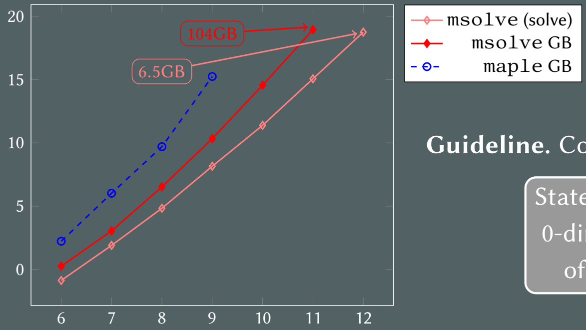
`https://msolve.lip6.fr`

`https://github.com/algebraic-solving/msolve`

SORBONNE UNIVERSITÉ

RPTU

Quantitative aspects



Guideline. Compute only what you need(!)

State-of-the-art handles 0-dimensional systems of degree $\simeq$ 10 000

plain C library

Berthomieu, Eder, Neiger, S.

$\simeq$ 55 000 lines, license GPLv2+

uses GMP and FLINT

https://msolve.lip6.fr

https://github.com/algebraic-solving/msolve

Degree bounds & complexity

There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

(**Bousquet-Mélou/Jéhanne**)

There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

(**Bousquet-Mélou/Jéhanne**)

Let δ be the degree of $\mathscr{P}$. Then, the degree of $\mathscr{A}$ is dominated by $\frac{\delta^{3k}}{k!}$.

Degree bounds & complexity

There exists a non-zero bivariate polynomial $\mathscr{A}$ s. t. $\mathscr{A}(G_a, t) \equiv 0$.

(**Bousquet-Mélou/Jéhanne**)

Let δ be the degree of $\mathscr{P}$. Then, the degree of $\mathscr{A}$ is dominated by $\frac{\delta^{3k}}{k!}$.

One can compute $\mathscr{A}$ using

$$\tilde{O}\left(\delta^{6k}\left(k^2 + \delta^{k+3} + \frac{\delta^{1.89k}}{k!}\right)\right)$$

arithmetic operations.

Geometric method

Projection ↕ **Elimination**

$$\begin{aligned} {\gamma_1}^2 - {\gamma_2}^2 &= 0 \\ \gamma_1 u - \gamma_2 &= 0 \\ \gamma_2 u - \gamma_1 &= 0 \\ u^2 - 1 &= 0 \\ \kappa &= 1 \end{aligned}$$

Projection ↕ **Elimination**

$$\begin{aligned} \gamma_1{}^2 - \gamma_2{}^2 &= 0 \\ \gamma_1 u - \gamma_2 &= 0 \\ \gamma_2 u - \gamma_1 &= 0 \\ u^2 - 1 &= 0 \\ \kappa &= 1 \end{aligned}$$

Projection on the (γ_1, γ_2, u)-space

Geometric method

Projection ↕ **Elimination**

$$\begin{aligned}\gamma_1{}^2 - \gamma_2{}^2 &= 0\\ \gamma_1 u - \gamma_2 &= 0\\ \gamma_2 u - \gamma_1 &= 0\\ u^2 - 1 &= 0\\ \kappa &= 1\end{aligned}$$

Almost the projection on the (γ_1, γ_2, u)-space

Extra ”hidden” condition:

$\gamma_2 \neq 0, \gamma_1 \neq 0$

Geometric method

Projection ↕ **Elimination**

$$\begin{array}{rcl} \boxed{\begin{array}{rcl} {\gamma_1}^2 - {\gamma_2}^2 & = & 0 \\ \gamma_1 u - \gamma_2 & = & 0 \\ \gamma_2 u - \gamma_1 & = & 0 \end{array}} \\ u^2 - 1 & = & 0 \\ \kappa & = & 1 \end{array}$$

Almost the projection on the (γ_1, γ_2, u)-space

Extra "hidden" condition:

$\gamma_2 \neq 0, \gamma_1 \neq 0$

☛ leading coefficients $= 0$

Geometric method

Projection ↕ **Elimination**

$$
\begin{array}{rcl}
\gamma_1{}^2 - \gamma_2{}^2 &=& 0\\
\gamma_1 u - \gamma_2 &=& 0\\
\gamma_2 u - \gamma_1 &=& 0\\
u^2 - 1 &=& 0\\
\kappa &=& 1
\end{array}
$$

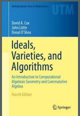
Almost the projection on the (γ_1, γ_2, u)-space

Extra "hidden" condition:

$\gamma_2 \neq 0, \gamma_1 \neq 0$

☛ leading coefficients $= 0$

The Elimination and Extension Theorems



Gröbner bases $\rightarrow$ **Triangular system**

Geometric method

Projection

↕

Elimination

$$
\begin{array}{rcl}
{\gamma_1}^2 - {\gamma_2}^2 & = & 0 \\
\gamma_1 u - \gamma_2 & = & 0 \\
\gamma_2 u - \gamma_1 & = & 0 \\
u^2 - 1 & = & 0 \\
\kappa & = & 1
\end{array}
$$

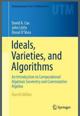
Almost the projection on the (γ_1, γ_2, u)-space

Extra "hidden" condition:

$\gamma_2 \neq 0, \gamma_1 \neq 0$

☛ leading coefficients $= 0$

The Elimination and Extension Theorems



Gröbner bases $\to$ **Triangular system**

Control on the cardinality of fibers counted with multiplicities

The univariate case

Take $f \in \mathbb{K}[u]$ of degree d
roots $\{\mu_1, \ldots, \mu_d\}$

$$V = \begin{bmatrix} 1 & \mu_1 & \cdots & \mu_1^{d-1} \\ \vdots & & & \vdots \\ 1 & \mu_d & \cdots & \mu_d^{d-1} \end{bmatrix}$$

$$V^T.V = \begin{bmatrix} 1 & N_1 & \cdots & N_{d-1} \\ N_1 & N_2 & & N_d \\ \vdots & \vdots & \ddots & \vdots \\ N_{d-1} & N_d & \cdots & N_{2d-2} \end{bmatrix}$$

Geometric method and root counting

The univariate case

Take $f \in \mathbb{K}[u]$ of degree d
roots $\{\mu_1, \ldots, \mu_d\}$

$$V = \begin{bmatrix} 1 & \mu_1 & \cdots & \mu_1^{d-1} \\ \vdots & & & \vdots \\ 1 & \mu_d & \cdots & \mu_d^{d-1} \end{bmatrix}$$

Hermite matrix

$$V^T.V = \begin{bmatrix} 1 & N_1 & \cdots & N_{d-1} \\ N_1 & N_2 & & N_d \\ \vdots & \vdots & \ddots & \vdots \\ N_{d-1} & N_d & \cdots & N_{2d-2} \end{bmatrix}$$

$N_i = \sum_{j=1}^{d} \mu_j^i$
polynomials in
the coefficients of f

Geometric method and root counting Bostan/Notarantonio/S.

The univariate case

Take $f \in \mathbb{K}[u]$ of degree d
roots $\{\mu_1, \ldots, \mu_d\}$

$$V = \begin{bmatrix} 1 & \mu_1 & \cdots & \mu_1^{d-1} \\ \vdots & & & \vdots \\ 1 & \mu_d & \cdots & \mu_d^{d-1} \end{bmatrix}$$

Hermite matrix

$$V^T.V = \begin{bmatrix} 1 & N_1 & \cdots & N_{d-1} \\ N_1 & N_2 & & N_d \\ \vdots & \vdots & \ddots & \vdots \\ N_{d-1} & N_d & \cdots & N_{2d-2} \end{bmatrix}$$

$N_i = \sum_{j=1}^{d} \mu_j^i$
polynomials in
the coefficients of f

The multivariate case

$\mathscr{P}_1 = \cdots = \mathscr{P}_s = 0$
in $\mathbb{Q}(\gamma_1)[u, \gamma_2, \ldots, \gamma_k]$
defining a 0-dimensional set

Multivariate generalization
of Hermite matrices

Conclusions and perspectives

DDESolver Maple package written by Hadrien Notarantonio

https://github.com/HNotarantonio/ddesolver

Example	k	time(dupl)	time(geo)
triangulation	2	55 secs	1 min. 10 secs
4-constellations	3	4 min.	41 secs.
4-tamari	3	2d. 2h.	6 min.

Conclusions and perspectives

DDESolver Maple package written by Hadrien Notarantonio

https://github.com/HNotarantonio/ddesolver

Example	k	time(dupl)	time(geo)
triangulation	2	55 secs	1 min. 10 secs
4-constellations	3	4 min.	41 secs.
4-tamari	3	2d. 2h.	6 min.

☛ **Not covered by this talk.** Combining guessing approaches with polynomial systems solvers

Conclusions and perspectives

DDESolver Maple package written by Hadrien Notarantonio

`https://github.com/HNotarantonio/ddesolver`

Example	k	time(dupl)	time(geo)
triangulation	2	55 secs	1 min. 10 secs
4-constellations	3	4 min.	41 secs.
4-tamari	3	2d. 2h.	6 min.

☛ **Not covered by this talk.** Combining guessing approaches with polynomial systems solvers

☛ Singularities of series depending with coefficients depending on a parameter

Bousquet-Mélou/Notarantonio

Conclusions and perspectives

DDESolver Maple package written by Hadrien Notarantonio

`https://github.com/HNotarantonio/ddesolver`

Example	k	time(dupl)	time(geo)
triangulation	2	55 secs	1 min. 10 secs
4-constellations	3	4 min.	41 secs.
4-tamari	3	2d. 2h.	6 min.

☛ **Not covered by this talk.** Combining guessing approaches with polynomial systems solvers

☛ Singularities of series depending with coefficients depending on a parameter

Bousquet-Mélou/Notarantonio

☛ Better algorithms for algebraic elimination?

$\rightsquigarrow$ Critical point structure of $\mathscr{P} = \frac{\partial \mathscr{P}}{\partial \kappa} = \frac{\partial \mathscr{P}}{\partial u} = 0$