Forme limite du tas de sable abélien avec fuite

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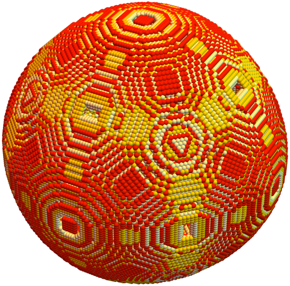
En commun avec Théo Ballu, Cédric Boutillier et Sevak Mkrtchyan

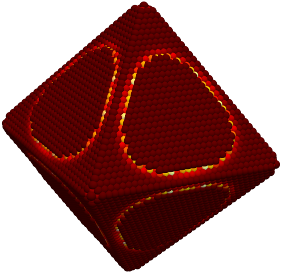
arXiv:2501.12664

4 décembre 2025

Séminaire Philippe Flajolet

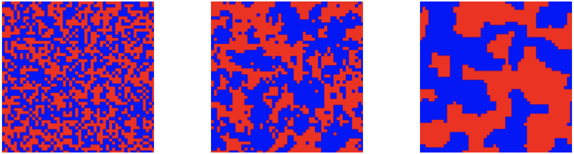
Examples of limit shapes



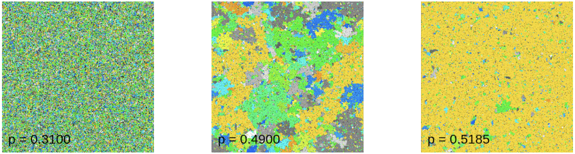


Self-organized criticality

Ising model



Bond percolation



Self-organized criticality

Critical point serves as an attractor: the parameters need not be fixed with precision, because the dynamical system essentially adjusts itself as it evolves toward criticality

Cellular automaton proposed by physicists to illustrate the notion of self-organized criticality

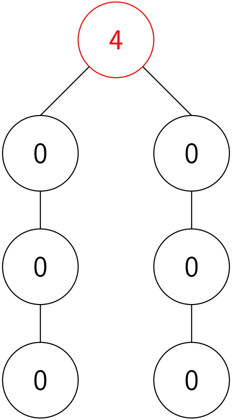
[Bak Tang Wiesenfeld '87] [Dhar '90]

The model

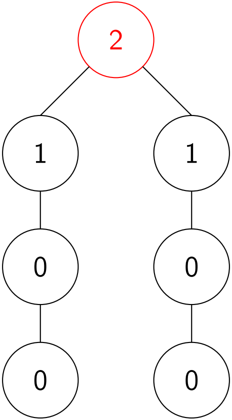
Let G be a graph

- On each vertex, put an integer number of grains of sand
- If a vertex has more grains of sand that its outdegree, then it is unstable and topples by sending one grain of sand to each of its neighbors
- Vertices topple until they all become stable

A simple example



A simple example



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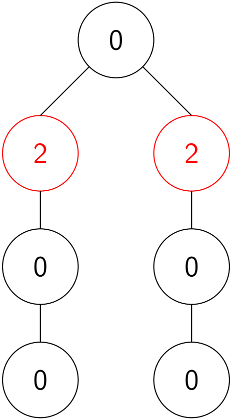
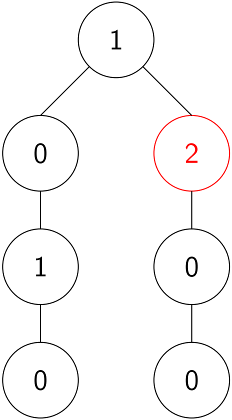
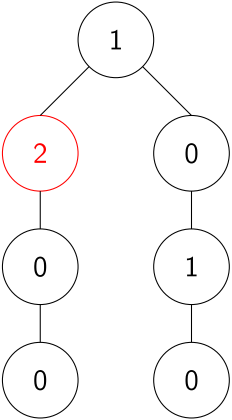


Figure: Two vertices are unstable

A simple example





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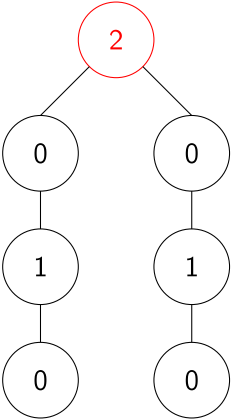
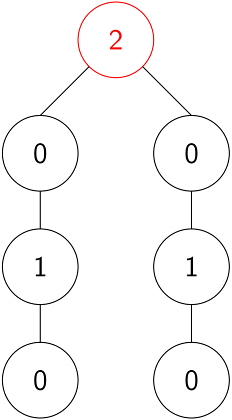




A simple example

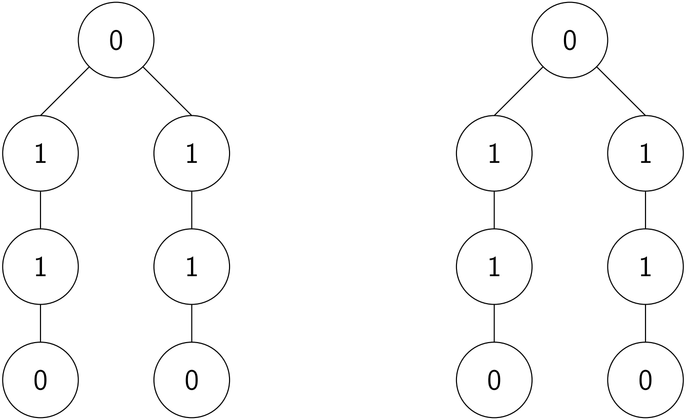
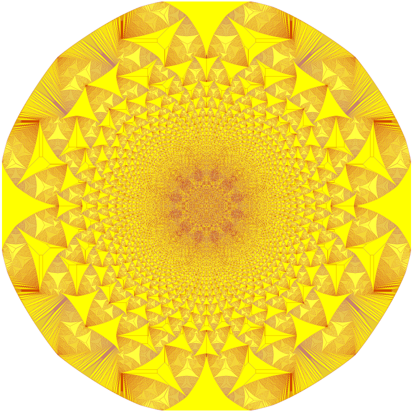


Figure: The order chosen to topple unstable vertices does not modify the eventual stable configuration (hence the name *abelian*)

A simulation by Sevak Mkrtchyan

On the grid $\mathbb{Z}^2$, put $N = 10^7$ grains of sand at the origin and let it topple



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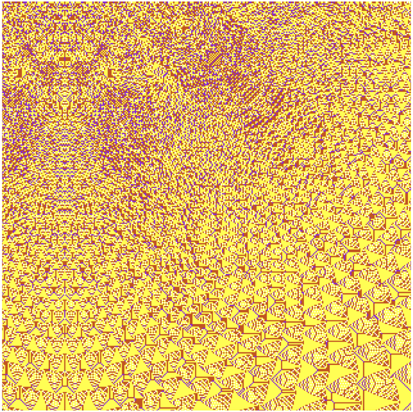


Known results and conjectures

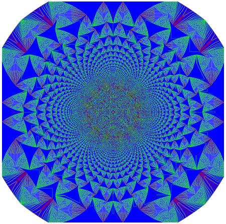
Known results

- Sandpile bounded between circles of radius $c\sqrt{N}$ and $C\sqrt{N}$,
- Existence limit shape (after normalization by $\sqrt{N}$),
- Fractal structure

[Pegden Smart '13] [Levine Pegden Smart '16]

Conjectures

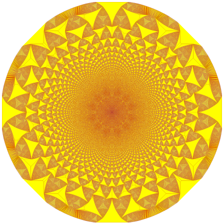
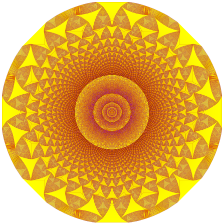
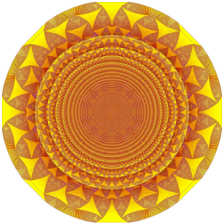
- Convexity of the limit shape
- Limit shape is not a disc
- Existence of flat areas



Main model in this talk

The leaky model

- Each time a vertex x topples, a proportion $\varepsilon_x \in (0, 1)$ of the sand that leaves x disappears
- The amount of sand is no longer an integer
- The amount of sand sent to neighbors is no longer uniform

Example of toppling

$$\varepsilon = \frac{1}{2}$$

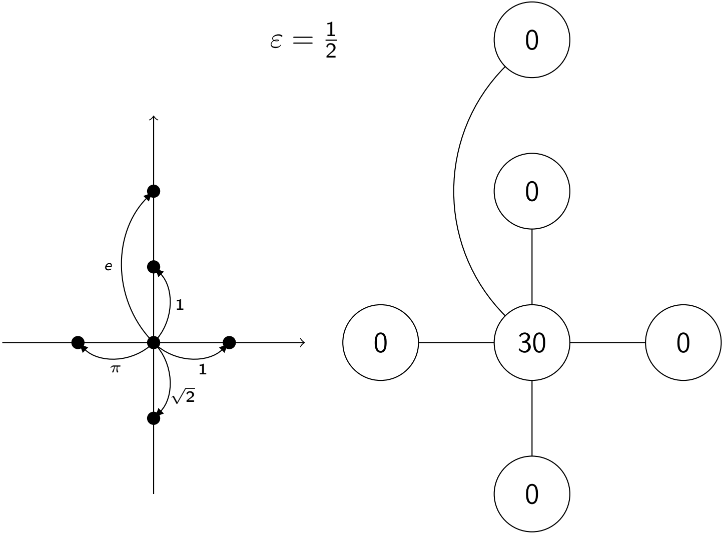


Figure: Left: amount of sand sent to neighbors. Right: initial configuration

Example of toppling

$\varepsilon = \frac{1}{2}$

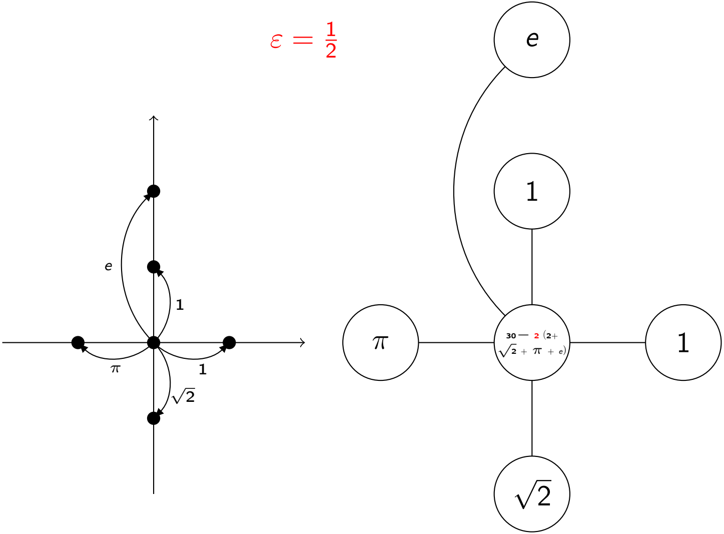
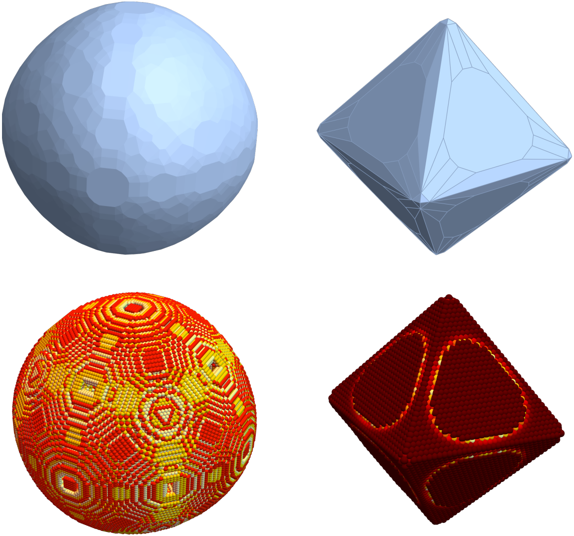


Figure: After the toppling

Simulations in $\mathbb{Z}^3$: various leakinesses



Main questions

We put N grains of sand at the origin and let the sandpile topple

- Is there a limit shape as $N \to \infty$? With which normalization? Still $\sqrt{N}$?
- What is the influence of the leakiness parameter ε?

Literature

Symmetric, nearest neighbors model in $\mathbb{Z}^2$

`[Alevy Mkrtchyan '22]`

Theorem 1.1. *Let $d > 1$ and $r = \log n - \frac{1}{2}\log\log n$. The boundary of the rescaled set of visited sites $r^{-1}D_{n,d}$ of the uniform Leaky-ASM converges as $n \to \infty$ to the dual of the boundary of the gaseous phase in the amoeba of the Laurent polynomial*

$$P(z,w) = \frac{4d - z - z^{-1} - w - w^{-1}}{4(d-1)}. \tag{1.1}$$

In this work

Generalization to graphs $\mathbb{Z}^d \times \{1, \ldots, p\}$ (thickenings of $\mathbb{Z}^d$)

Two ways of thinking of the graph

$\mathbb{Z}^d$ where each site appears in p colors or p layers of $\mathbb{Z}^d$

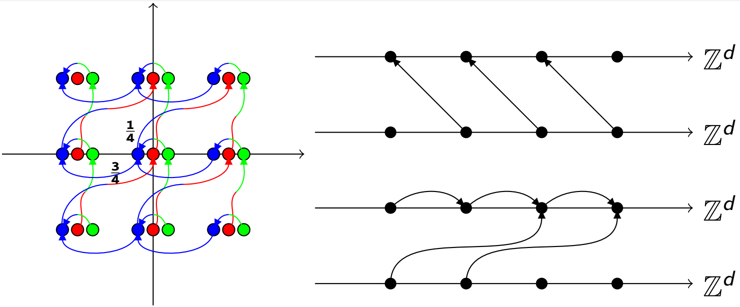


Figure: Left: $d = 2$ et $p = 3$. Right: $d = 1$ et $p = 4$

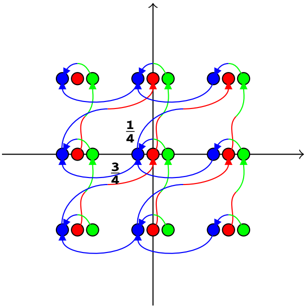
Markov additive process on $\mathbb{Z}^d \times \{1, \ldots, p\}$

Transition probability invariant along the $\mathbb{Z}^d$ component:

$$\forall x, x' \in \mathbb{Z}^d, \quad \forall i, j \in A, \quad \mathbb{P}\big((x,i) \to (x',i')\big) = \mathbb{P}\big((0,i) \to (x'-x, i')\big)$$

Two ways of thinking of the graph

$\mathbb{Z}^d$ where each site appears in p colors or p layers of $\mathbb{Z}^d$



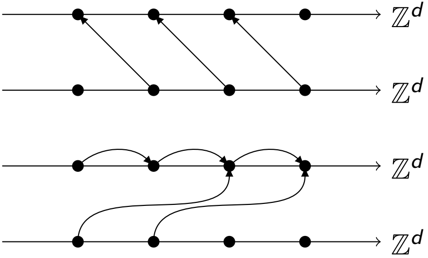
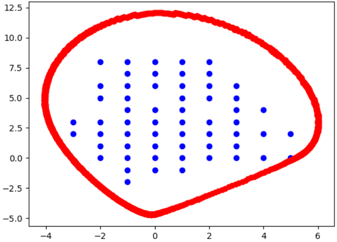
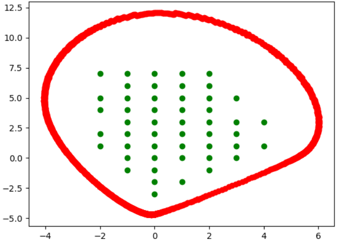


Figure: Left: $d = 2$ et $p = 3$. Right: $d = 1$ et $p = 4$

Assumptions on the sandpile

- Different leakiness parameters: ε_i for each of $i \in \{1, \dots, p\}$
- No symmetry of the model, no finite range hypothesis





Same limit shape on each of the p layers

Killed random walk

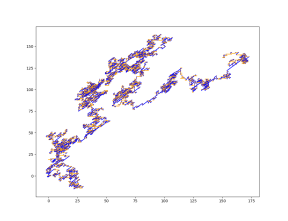
Definition

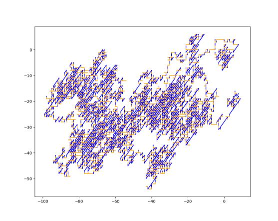
Killed random walk (X_n) on G naturally associated with the model.

At a vertex x:

- It is killed with probability ε_x
- If not killed, it jumps to y with probability proportional to the amount of sand x sends to y each time it topples

Simulations for $d = p = 2$





Two different behaviors (non-centered and centered)

The Green function

Transience of the random walk implies the finiteness of the Green function (0 is an origin)

$$G(0,x) := \sum_{n=0}^{\infty} \mathbb{P}_0\left(X_n = x\right) = \mathbb{E}_0\left[\text{number of visits of } x\right]$$

Of key importance: asymptotics of $G(0,x)$ as $x \to \infty$

The odometer function

$u(x) =$ total amount of sand sent and lost by x during topplings

Final configuration

A vertex x is said to be in the final configuration if $u(x) > 0$

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Link between the shape of the sandpile and Green function

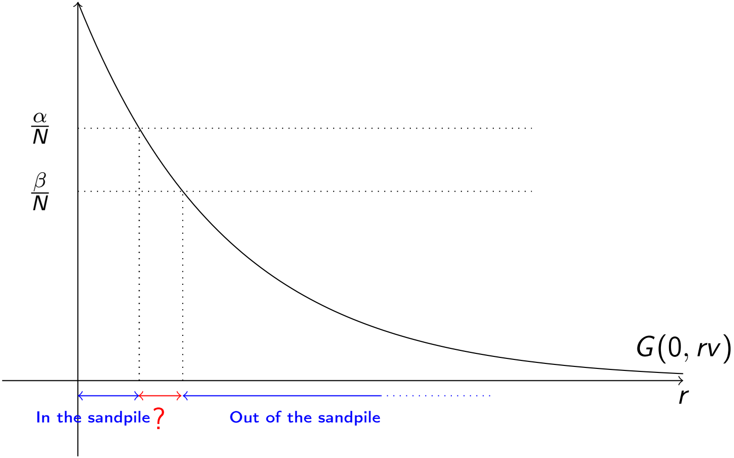
There exist two constants $\alpha, \beta > 0$ such that:

- If $G(0,x) > \frac{\alpha}{N}$, then x is in the final configuration
- If $G(0,x) < \frac{\beta}{N}$, then x is not in the final configuration

[Ballu Boutillier Mkrtchyan R '25]

Idea of the proof

Harmonicity properties of the odometer and Green functions



In a fixed direction v, if the Green function decreases fast enough, the "?" area is small enough to disappear as $N \to \infty$

Asymptotics of the Green function

Assume the process irreducible, aperiodic, non-centered, non-killed and with finite exponential moments of all orders. Then Ornstein-Zernike behavior as $\|x\| \to \infty$

$$G\big((0,i),(x,j)\big) \sim C_{i,j}(\widehat{x})\|x\|^{-\frac{d-1}{2}} e^{-\gamma(\widehat{x})\cdot x}$$

where:

- $\widehat{x} = \frac{x}{\|x\|}$
- $C_{i,j}$ are continuous, positive functions defined on $\mathbb{S}^{d-1}$
- γ is a homeomorphism between $\mathbb{S}^{d-1}$ and the boundary of a convex compact subset of $\mathbb{R}^d$; does not depend on i and j

Moreover, when x is not in the direction of the drift, $\gamma(\widehat{x}) \cdot x > 0$

[Dussaule '20] [Ballu '24]

Convergence result when $N \to \infty$

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Asymptotics of the Green function for the killed random walk

A Doob transform turns the killed random walk into a non-centered, non-killed one, allowing to use the previous asymptotics:

$$G\big((0,i),(x,j)\big) \sim c_{i,j}\left(\widehat{x}\right)\|x\|^{-\frac{d-1}{2}} e^{-\gamma(\widehat{x})\cdot x}$$

Main theorem

The shape of the sandpile converges for the Hausdorff distance, once normalized by $\log N$. The limit shape is described in spherical coordinates by

$$\left\{ \frac{v}{\gamma(v)\cdot v} \mid v \in \mathbb{S}^{d-1} \right\}$$

Same limit shape on each of the p layers

[Ballu Boutillier Mkrtchyan R '25]

Let $(E_N)_{N\geqslant 0}$ be subsets of $\mathbb{R}^d$ and

$$\mathcal{C} = \left\{ r_u u \mid u \in \mathbb{S}^{d-1} \right\}$$

with $r : u \in \mathbb{S}^{d-1} \mapsto r_u \geqslant 0$

Convergence of sets

The limit shape of the sets E_N is delimited by $\mathcal{C}$ if, for every $u \in \mathbb{S}^{d-1}$, there exist $(r_{u,N})_{N\geqslant 0}$ and $(R_{u,N})_{N\geqslant 0}$ such that

$$\left\{ ru \mid u \in \mathbb{S}^{d-1}, r \leqslant r_{u,N} \right\} \subset E_N \subset \left\{ Ru \mid u \in \mathbb{S}^{d-1}, R \leqslant R_{u,N} \right\}$$

and for every $u \in \mathbb{S}^{d-1}$

$$r_{u,N} \xrightarrow[N\to\infty]{} r_u \quad \text{and} \quad R_{u,N} \xrightarrow[N\to\infty]{} r_u$$

Uniform convergence

The convergence is uniform if the two limits are uniform in u

$$\sup_{u \in \mathbb{S}^{d-1}} |r_{u,N} - r_u| \xrightarrow[N\to\infty]{} 0 \quad \text{and} \quad \sup_{u \in \mathbb{S}^{d-1}} |R_{u,N} - r_u| \xrightarrow[N\to\infty]{} 0$$

Uniform convergence $\Rightarrow$ classical Hausdorff distance convergence

Hausdorff convergence

Hausdorff distance between non-empty $X, Y \subset \mathbb{R}^d$

$$d_H(X, Y) = \max\left(\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right)$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$

Limit shape by convex duality

Dual convex

Let K be a convex, compact subset of $\mathbb{R}^d$ such that $0 \in \mathring{K}$. Let $x \in \partial K$ such that the hyperplane of equation

$$n \cdot y = 1$$

is tangent to K at x. Define $x^* = n$ (if there are several tangent hyperplanes at x, x^* is a set)

$$\bigcup x^* = \text{boundary of the dual convex of } K$$

Limit shape

Denote $\rho \circ L(x)$ the spectral radius of the Laplace transform of the killed Markov additive process

$$\text{Limit shape of the sandpile} = \text{dual convex of } (\rho \circ L)^{-1}([0,1])$$

`[Ballu Boutillier Mkrtchyan R '25]`

Two examples with $p = 1$

Limit shape

Denote $L(x)$ the Laplace transform of the killed Markov additive process

$$\text{Limit shape of the sandpile} = \text{dual convex of } L^{-1}([0,1])$$

Level line of the Laplace transform

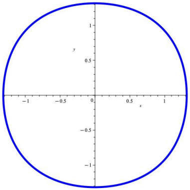
Link with the enumeration of walks and the kernel method

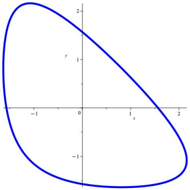
The simple random walk

$$\frac{1}{4}\left(e^{x} + e^{-x} + e^{y} + e^{-y}\right)$$

Kreweras walk

$$\frac{1}{3}\left(e^{-x} + e^{-y} + e^{x+y}\right)$$





First limit regime: killing goes to 0

Setting

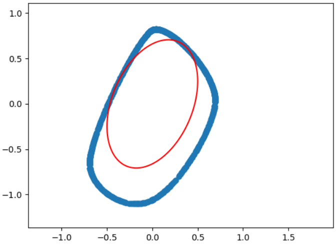
We assume $\varepsilon_1 = \ldots = \varepsilon_p =: \varepsilon \to 0$ after $N \to \infty$

Result

After normalization by $\sqrt{\varepsilon}$, the limit shape tends to the ellipsoid

$$\left\{x \in \mathbb{R}^d \mid 2x^\top \sigma x \leqslant 1\right\}$$

where σ is the energy matrix ($\simeq$ covariance matrix) of the non-killed random walk



Likely a discontinuity at $\varepsilon = 0$ since the ellipsoid has no flat areas

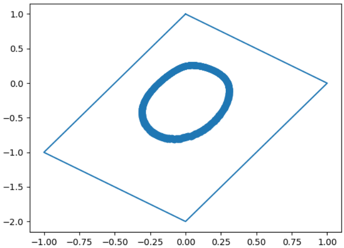
Proof strategy

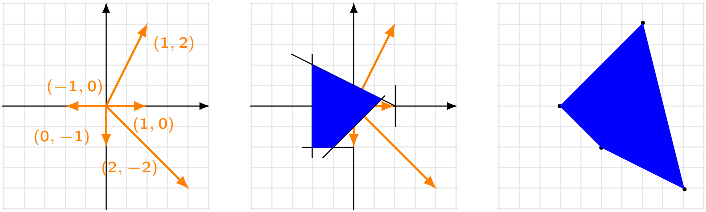
Study the Laplace transform and use convex duality

Second limit regime: killing goes to 1

Theorem

When $\varepsilon \to 1$, after normalization by $-\log(1-\varepsilon)$, the limit shape tends to a polytope





Left picture: the step set. Middle picture: the limit level set. Right picture: the Newton polytope associated with the step set

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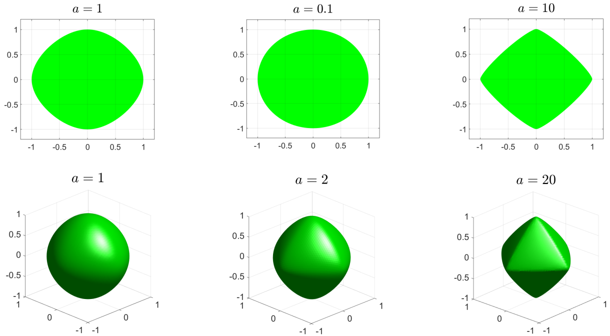
Asymptotics of the Green function for the simple random walk

Killing rate $\varepsilon \approx \frac{a}{1+a}$ [Michta Slade '22]

fixed $a > 0$	anisotropic OZ	$m_a^{(d-3)/2} \lvert x \rvert_a^{-(d-1)/2} e^{-m_a \lvert x \rvert_a}$
$a\lVert x \rVert_2 \to \infty,\ a^3 \lVert x \rVert_2 \to 0$	isotropic OZ	$a^{(d-3)/2} \lVert x \rVert_2^{-(d-1)/2} e^{-\sqrt{2d}a \lVert x \rVert_2}$
fixed $a\lVert x \rVert_2 > 0$	massive continuum	$\lVert x \rVert_2^{-(d-2)}$
$a = 0 \ \ (d > 2)$	massless continuum	$\lVert x \rVert_2^{-(d-2)}$



Link with large deviation theory

Random walk with Laplace transform $L(x) = \mathbb{E}(e^{x \cdot X_1})$

(First) rate function

Legendre transform $\Lambda(u) = \sup_{x \in \mathbb{R}^d} \{x \cdot u - \log L(x)\}$

Large deviation principle

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{X_n}{n} \in V_\varepsilon(u)\right) = -\Lambda(u)$$

(Second) rate function

$$D(u) = \inf_{s>0} \frac{\Lambda(su)}{s}$$

[Borovkov Mogulskii '96]
[Korshunov '96]

The link

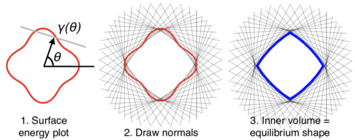
$$\gamma(\widehat{x}) = D(\widehat{x})$$

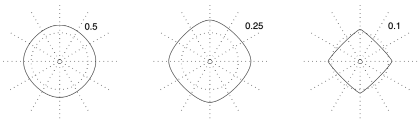
[Borovkov Mogulskii '96]
[Messikh '06]

Dual curves and Wulff crystals

Wulff shape in the Ising model

- Equilibrium macroscopic shape of a large droplet of one phase inside the other
- Minimizes the anisotropic surface tension of interfaces
- Boundary diamond-like shape at low temperature and more circular near critical point [Cerf Messikh '10]





Two-point correlation function

Green function

[Messikh '06]

A link

$\gamma(\widehat{x}) =$ dual of $L^{-1}(\{1\})$

[Borovkov Mogulskii '96]

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Context: homogeneous non-centered case in $\mathbb{Z}^d$

- $X_n = Y_1 + \cdots + Y_n$, where $(Y_k)_{k\geqslant 1}$ are iid random variables
- $\mathbb{E}[Y_1] \neq 0$ (transient case)

Theorem

If Laplace transform L finite in a neighbourhood of $L^{-1}(\{1\})$,

$$G(0,x) = \sum_{n=0}^{\infty} \mathbb{P}_0\left(X_n = x\right) \sim c(\widehat{x})\|x\|^{-\frac{d-1}{2}} \exp\left(-\gamma(\widehat{x})\cdot x\right)$$

where $\widehat{x} = \frac{x}{\|x\|}$ (Ornstein-Zernike behavior) [Ney Spitzer '66]

Ideas of proof

- Local limit theorem [Ney Spitzer '66]
- Integral expression for the Green function [Woess '00]

Various extensions

Almost homogeneous random walks

Homogeneous except on a finite number of vertices of the plane

[Kurkova Malyshev '98]

Random walks in cones

Killed or reflected random walks in half spaces, quarter planes and cones

[Ignatiuk] [Denisov Wachtel] [Kurkova R] [Mustapha]

Markov additive processes

State space $E = \mathbb{Z}^d \times A$ [Dussaule '20] [Ballu '24]

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The homeomorphism

The function γ

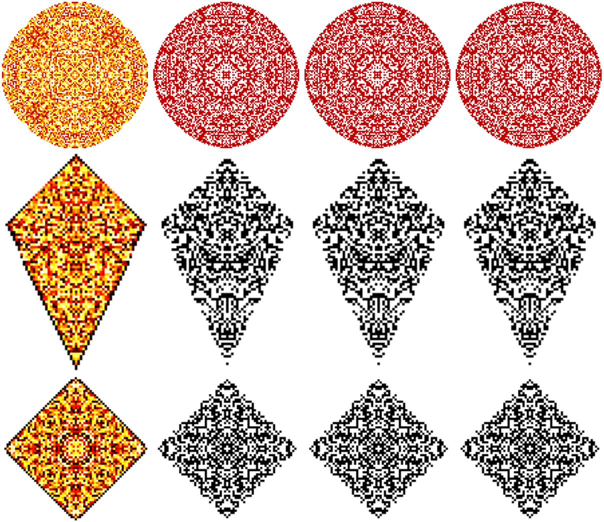
$$\begin{array}{rcl} \left\{x \in \mathbb{R}^d \mid \rho \circ L(x) = 1\right\} & \longrightarrow & \mathbb{S}^{d-1} \\ x & \longmapsto & \frac{\nabla(\rho \circ L)(x)}{\|\nabla(\rho \circ L)(x)\|} \end{array}$$

is a homeomorphism [Hennequin '63]

Doob transform

Surjectivity allows to choose the direction of the drift in the Doob transform

Merci pour votre attention!



Merci pour votre attention!

Doob transform

The Doob transform

If $\rho \circ L\mu(c) = 1$ and $\varphi_c \in (\mathbb{R}_+^*)^d$ is a Perron-Frobenius eigenvector, then the matrix of measures

$$(\mu_c)_{i,j}(x) = \frac{(\varphi_c)_j}{(\varphi_c)_i} e^{c\cdot x} \mu_{i,j}(x)$$

defines a Markov additive process. It is the Doob transform of parameter c

Drift and Hennequin homeomorphism

$$\mathbb{E}[\mu_c] = \nabla(\rho \circ L)(c) \neq 0$$

Green functions

$$G_c\big((0,i),(x,j)\big) = c_{i,j} e^{c\cdot x} G\big((0,i),(x,j)\big)$$