Progressive and rushed Dyck paths

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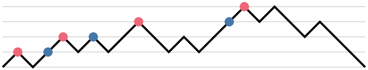
Progressive and rushed paths

Definition

A Dyck path is progressive if every visit at height $h \geq 2$ is preceded by at least two visits at height $h-1$.



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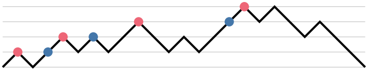

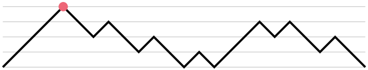

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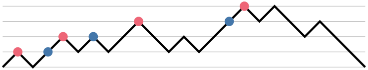
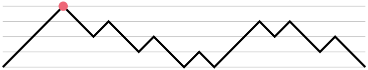


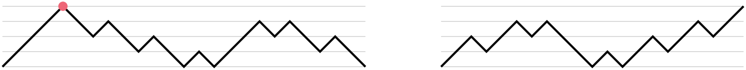


Theorem [Asinowski and Jelínek, A287709]

There are as many progressive as rushed paths of length $2n$.

Rushed paths and culminating paths

- A rushed path of height h and length $2n$ is equivalent to a culminating path ending at $(2n-h, h)$.
  [Bousquet-Mélou and Ponty, 2008]



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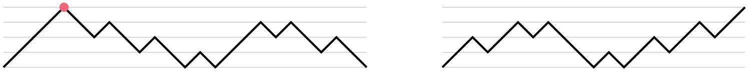

- The generating function of rushed paths is:

$$R(z) = \sum_{h=1}^{\infty} \frac{z^h}{F_h(z)}$$

where $F_h(z)$ is the Fibonacci polynomial.

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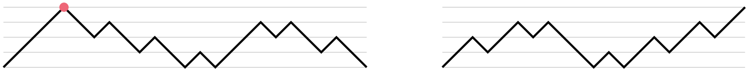

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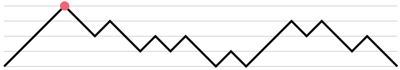
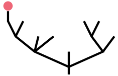
- Rushed and culminating paths of length n are different.

One-sided trees and asymptotics





Theorem **[Durhuus and Ünel, 2023]**

The number of one-sided trees (=rushed paths) satisfies:

$$r_n \sim \lambda 4^n e^{-\nu n^{1/3}} n^{-5/6}.$$

Their height satisfies:

$$\frac{h(R_n) - \mu n^{1/3}}{\sigma n^{1/6}} \xrightarrow{d} \mathcal{N}(0,1).$$

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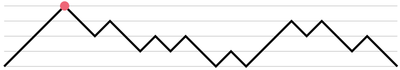

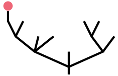

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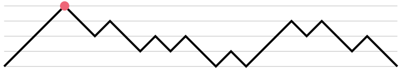

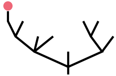

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- Dyck paths with weight 2^{-h} have the same asymptotics.
  [Beaton and McKay, 2014; Durhuus and Ünel, 2023]

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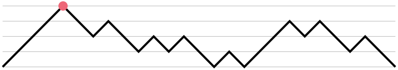

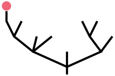

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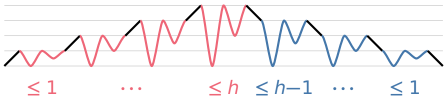
- Dyck paths with weight 2^{-h} have the same asymptotics. **[Beaton and McKay, 2014; Durhuus and Ünel, 2023]**
- Durhuus and Ünel give the local limit of one-sided trees.

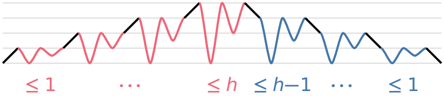
Dyck paths ⟷ progressive culminating paths

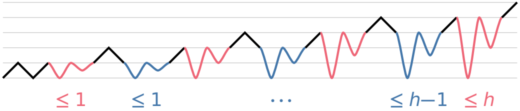
- Start from a Dyck path of length $2n$ and height h and decompose it into downward factors.



Dyck paths ⟷ progressive culminating paths

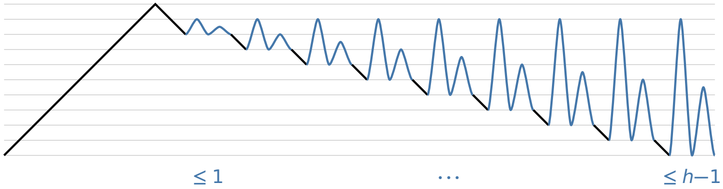
- Start from a Dyck path of length $2n$ and height h and decompose it into downward factors.
- Rearrange the factors into a progressive culminating path ending at $(2n + h + 1, h + 1)$.





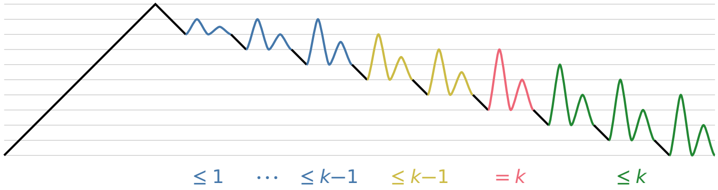
Rushed paths ⟷ progressive paths

- Decompose the rushed path into upward factors.



Rushed paths ⟷ progressive paths

- Decompose the rushed path into upward factors.
- Find the leftmost highest factor



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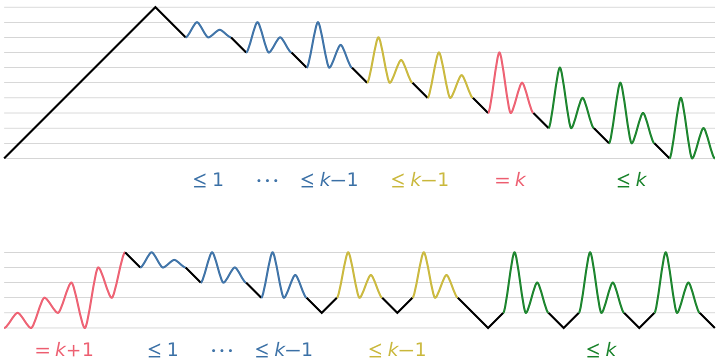
- Decompose the rushed path into upward factors.
- Find the leftmost highest factor and use the first bijection.



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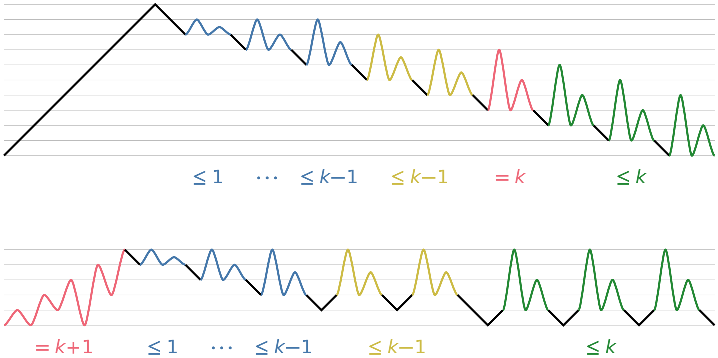
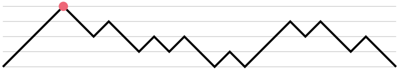


- The resulting progressive path has equal or lower height.

Asymptotics of rushed paths

[Durhuus and Ünel, 2023]



- Start from:

$$r_{n,h-1} = \frac{4^{n+1}}{2^h h} \sum_{j=1}^{\left\lfloor \frac{h-1}{2} \right\rfloor} (-1)^{j+1} \sin^2 \frac{j\pi}{h} \cos^{2n-h} \frac{j\pi}{h}$$

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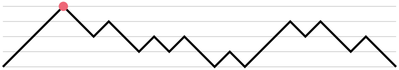

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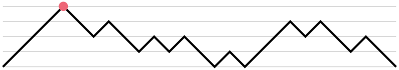

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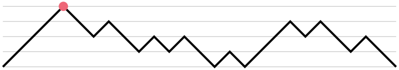

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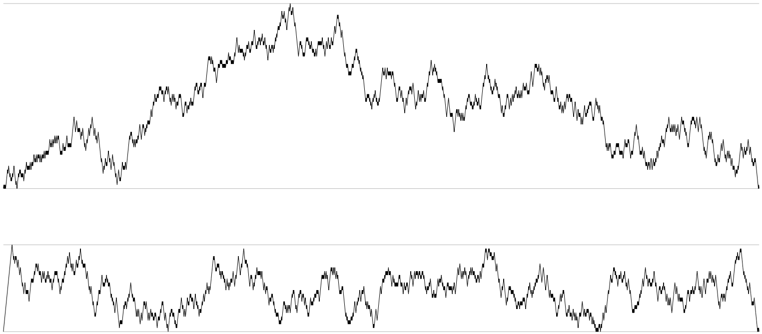
Arches of rushed paths



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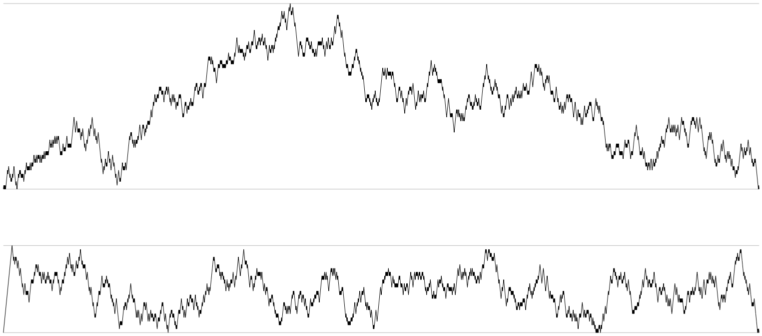
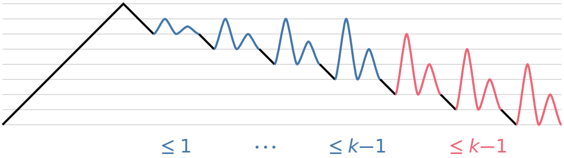


- Durhuus and Ünel's local limit of one-sided trees encodes the arch decomposition of rushed paths.
- The number of arches tends to a discrete law:

$$\mathbf{E}\left[z^{a(R)}\right] \to \frac{z}{2-z} \, 2^{\frac{z}{2-z}} - 1 \, .$$

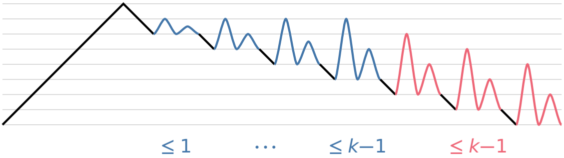
Height of progressive paths

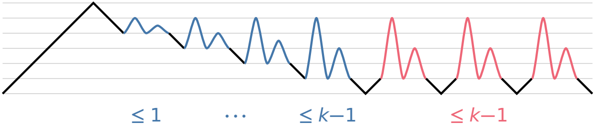
- Take (R, P) a rushed/progressive pair and assume $h(P) \leq k$.



Height of progressive paths

- Take (R,P) a rushed/progressive pair and assume $h(P) \leq k$.
- Send it bijectively to a rushed path of length $2n+2$, height $k+1$ and $h(R)-k+1$ arches.





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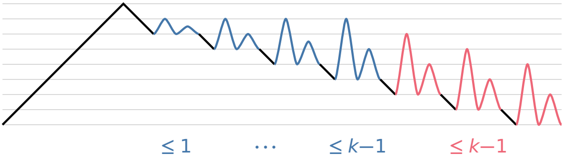
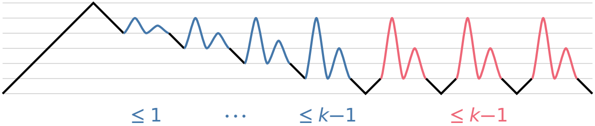
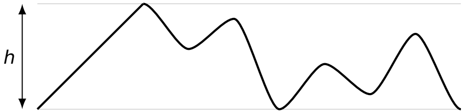




- From the local limit, we get:

$$\mathbf{E}\left[z^{h(R)-h(P)}\right] \to \frac{1}{z}\left(1-\frac{1-z}{2-z}\,2^{\frac{2}{2-z}}\right).$$

Random sampling of rushed paths



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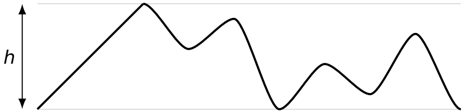
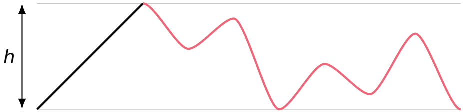


- Efficient random sampling of culminating paths is done in **[Bousquet-Mélou and Ponty, 2008]** but is not applicable.
- Instead, the strategy is:
  - select a height h with the right distribution,
  - sample a rushed path of height h with the recursive method.

Selecting the height



- To select the height, we need $r_h = \left[z^{n-h}\right]\frac{1}{F_h(z)}$, $h \leq n$.

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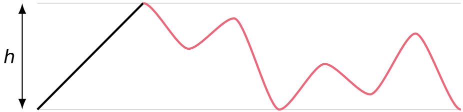

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- We use a bound on $\sum_{h=m}^{n} r_h$ to compute the r_h on demand.

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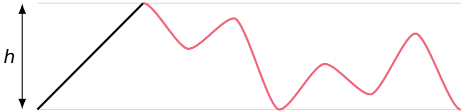

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- We compute intervals containing r_h, refined on demand.

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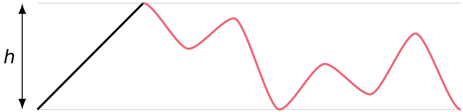

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- Exact uniformity is still guaranteed.

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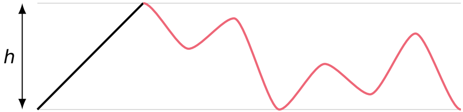

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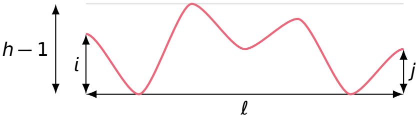
- We use a bound on $\sum_{h=m}^{n} r_h$ to compute the r_h on demand.
- We compute intervals containing r_h, refined on demand.
- Exact uniformity is still guaranteed.
- Average complexity seems to be $\mathrm{O}\left(n^{1/3}\log(n)\,\mathrm{M}(n^{2/3})\right)$.

Drawing the path



- To draw the path, we need $d_{h,i,j,\ell} = \left[z^{\frac{\ell-j+i}{2}}\right]\frac{F_i(z)F_{h-1-j}(z)}{F_h(z)}$.

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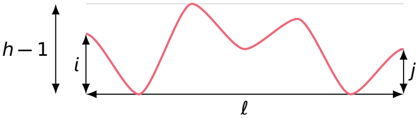

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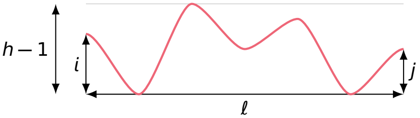

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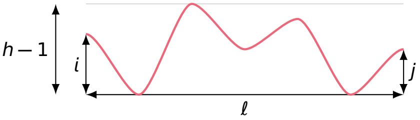
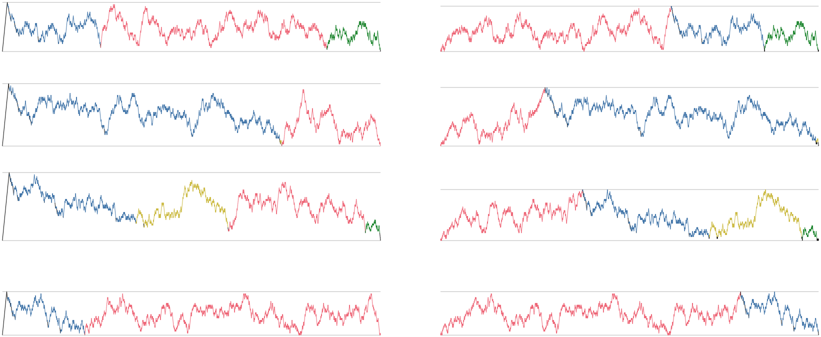
- Divide and conquer: we compute only $\ell = \frac{n-h-1}{2}, \frac{n-h-1}{4}, \ldots$
- Complexity $\mathrm{O}\left(\log(n)\,\mathrm{M}(n^{4/3}) + n^{5/3}\right)$.

Drawing the path



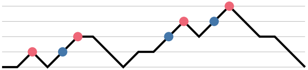
- To draw the path, we need $d_{h,i,j,\ell} = \left[z^{\frac{\ell-j+i}{2}}\right]\frac{F_i(z)F_{h-1-j}(z)}{F_h(z)}$.
- Divide and conquer: we compute only $\ell = \frac{n-h-1}{2}, \frac{n-h-1}{4}, \ldots$
- Complexity $\mathrm{O}\left(\log(n)\,\mathrm{M}(n^{4/3}) + n^{5/3}\right)$.
- Total complexity subquadratic with fast computer algebra.
  [Bostan and Mori, 2021; van der Hoeven, 2008]

Random rushed path examples



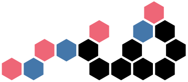
Perspectives

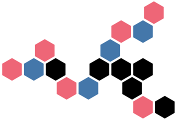
- Paths with different steps?



- Doubly progressive paths?

- Directed animals?





- Compositions with number of parts divisible by first part have g.f. $\left(1+\frac{z^2}{1-2z}\right)R\left(z-z^2\right)$. What's going on?