Count of metrics on combinatorial maps and Masur-Veech volumes

with E. Duryev and I. Yakovlev

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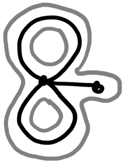
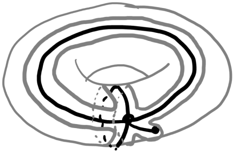
Metrics on maps

Counting metrics on maps

Maps / Ribbon graphs :



$\mathcal{RG}_{0,3}^{[5,1]}$



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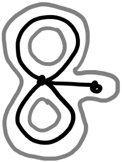

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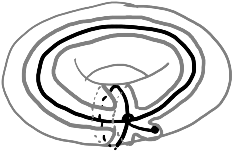


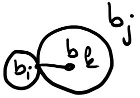
$\mathcal{RG}_{1,1}^{[5,1]}$

(Integer) metric on a map: length $\ell_i (\in \mathbb{Z}_+)$ on each edge i.

Question: How many metrics can be put on a ribbon graph with given face lengths?

Counting functions

Count of metrics on a ribbon graph with given face lengths:



$$\begin{pmatrix} b_i \\ b_j \\ b_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}$$

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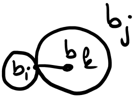

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More generally $P_G(\underline{b}) = \{\underline{\ell} \in \mathbb{R}_+^{E(G)}, \quad \underline{b} = A\underline{\ell}\}$ is a rationnal convex polytope.

$$F_G(b_1, \ldots, b_n) = \mathrm{card}\{\text{int. metrics on } G \text{ s.t. face lengths are } b_1, \ldots, b_n\}$$
$$= \mathrm{card}\{P_G(\underline{b}) \cap \mathbb{Z}_+^{E(G)}\} \quad \text{is a piecewise quasi-polynomial.}$$

$V_G(\underline{b}) = \mathrm{vol}(P_G(\underline{b}))$ is a piecewise polynomial.

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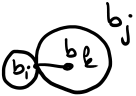

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$$\begin{aligned} F_G(b_1, \dots, b_n) &= \mathrm{card}\{\text{int. metrics on } G \text{ s.t. face lengths are } b_1, \dots, b_n\} \\ &= \mathrm{card}\{P_G(\underline{b}) \cap \mathbb{Z}_+^{E(G)}\} \quad \text{is a piecewise quasi-polynomial.} \end{aligned}$$

$V_G(\underline{b}) = \mathrm{vol}(P_G(\underline{b}))$ is a piecewise polynomial.

Outside the walls ($\sum_I b_i = \sum_J b_j$) and for higher degree terms:

$$2\, V_{g,n}^{k}(\underline{b}) := 2 \sum_{G \in \mathcal{RG}_{g,n}^{k}} \frac{V_G(\underline{b})}{|\,\mathrm{Aut}(G)|} \sim F_{g,n}^{k}(\underline{b}) := \sum_{G \in \mathcal{RG}_{g,n}^{k}} \frac{F_G(\underline{b})}{|\,\mathrm{Aut}(G)|}$$

Kontsevich polynomials

From now on $\underline{k} = (k_1, \ldots, k_r)$ is odd.

Definition (Blackbox)

$$N_{g,n}^{\underline{k}}(b_1, \ldots, b_n) = \frac{1}{2^{5g-6+2n-2M}} \sum_{\underline{d} \vdash 3g-3+n-M} \frac{\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_{\underline{k}}}{d_1! \ldots d_n!} b_1^{2d_1} \ldots b_n^{2d_n},$$

with $M = \sum (k_i - 3)/2$.

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Theorem (Kontsevich '92)

Outside of the walls, for integer $\underline{b} = (b_1, \ldots b_n)$ *such that* $\sum b_i$ *is even,*

$$N^{\underline{k}}_{g,n}(\underline{b}) = F^{\underline{k}}_{g,n}(\underline{b}) + l.d.t.$$

How to compute $\langle \tau_{\underline{d}} \rangle_{\underline{k}}$ and $N_{g,n}^{\underline{k}}$?

Generating series (writing $\underline{k} = ((2i+1)^{m_i})$):

$$F(t_*, s_*) = \sum_{n_*, m_*} \langle \prod \tau_i^{n_i} \rangle_{\underline{k}} \prod \frac{t_i^{n_i}}{n_i!} \prod s_j^{m_j} \text{ and } Z = \exp(F).$$

- At $\hat{s} = (0, 1, 0, 0, \dots)$, $F(t_*, \hat{s}) =: F(t_*)$ is the generating series for the "classical" $\langle \tau_{\underline{d}} \rangle$ ($\exp F(t_*)$ is a τ-function for the KdV-hierarchy).
- Coefficients $\langle \tau_{\underline{d}} \rangle_{\underline{k}}$ are obtained as $s-$ *derivatives* of F at $\hat{s}$.
- Kontsevch '92: Z is an asymptotic expansion of a matrix integral
- Di Francesco-Itzykson-Zuber '93:

$$\exists I : \mathbb{Q}\left[\frac{\partial}{\partial t_0}, \dots\right] \to \mathbb{Q}\left[\frac{\partial}{\partial s_0}, \dots\right] \text{ linear isomorphism s.t.}$$

$$\forall P \in \mathbb{Q}\left[\frac{\partial}{\partial t_0}, \dots\right], P(Z)_{|s_* = \hat{s}} = I(P) Z_{|s_* = \hat{s}}$$

Example: $N_{0,3}^{[5,1]}(b_1, b_2, b_3)$

Number of integer metrics on ribbon graphs with

- genus 0
- one vertex of valency 5 and one vertex of valency 1
- 3 faces of lengths b_1, b_2, b_3 ($\sum b_i$ even)

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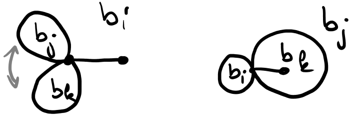
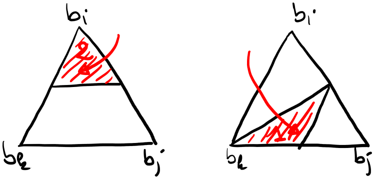


Contribution in simplex $b_i > 0$



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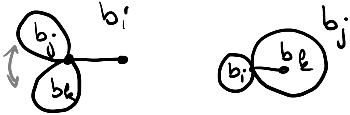

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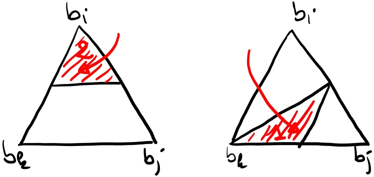
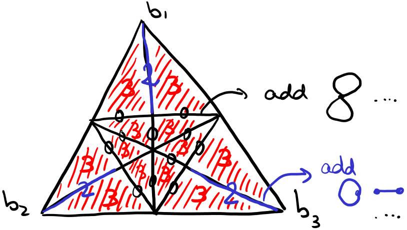


Total



$$N_{0,3}^{[5,1]}(b_1, b_2, b_3) = 3$$

Completion coefficients

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Theorem (Duryev-G-Yokovlev, 2025)

$$N^{\underline{\kappa},unlab}_{g,n}(\underline{b}) = 2 \cdot V^{\underline{\kappa}}_{g,n}(\underline{b}) + \sum_{m\geq 1} \sum_{\substack{g_0+g_1+\dots+g_m=g \\ g_i\geq 0}} \sum_{\substack{A_0\sqcup\dots\sqcup A_m=\{1,\dots,p\} \\ A_i\neq\emptyset}} \sum_{\epsilon_1,\dots,\epsilon_p\in\{0,1\}} \sum_{a:\{1,\dots,m\}\to\{1,\dots,\ell(\underline{\kappa})\}}$$

$$\frac{1}{m!\cdot 2^{m+|A_0|}} \cdot \frac{|\operatorname{Aut}(\underline{\kappa}^0)|}{|\operatorname{Aut}(\underline{\kappa})|} \cdot C_{\underline{g},\underline{A},\underline{\epsilon},a} \cdot 2 \cdot V^{\underline{\kappa}^0}_{g_0,n_0}(\underline{b}_0) \cdot \prod_{i=1}^{m} V^{[4g_i-2+2n_i^\bullet+2n_i^\circ]}_{g_i,(n_i^\bullet,n_i^\circ)}(\underline{b}_i^\bullet, \underline{b}_i^\circ), \tag{1}$$

where

$$C_{\underline{g},\underline{A},\underline{\epsilon},a} = \prod_{i=1}^{\ell(\underline{\kappa}^0)} \left(\kappa_i^0 \cdot \prod_{j\in a^{-1}(i)} (2g_j - 1 + n_j^\bullet + n_j^\circ) \cdot \frac{(\kappa_i-2)!!}{(\kappa_i - 2|a^{-1}(i)|)!!} \right). \tag{2}$$

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Explicit formula for the difference in terms of counting functions of degenerate ribbon graphs

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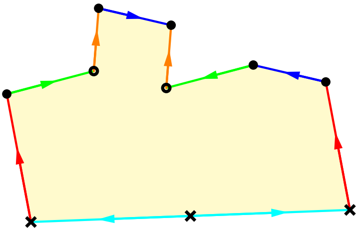
Theorem (Duryev-G-Yokovlev, 2025)

Explicit formula for the difference in terms of counting functions of degenerate ribbon graphs

Example: $N^{[5,1]}_{0,3} = F^{[5,1]}_{0,3} + F^{[2]}_{0,2} F^{[1,1]}_{0,1} + F^{[4]}_{0,3}$

Masur-Veech volumes

Quadratic differentials/ Half-translation surfaces



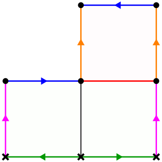
Half-translation surface
Flat metric
Conical angles $k \cdot \pi$

$\updownarrow$

Riemann surface
with a quadratic differential
(at most simple poles)
singularities of order
$k-2 \geq -1$

Ex: half-translation square-tiled surface of genus 1.

$$\sum (k_i - 2) = 4g - 4$$



Notations and definitions

$\underline{k} = (k_1, \dots, k_r)$ with k_i odd positive integer.

$\mathcal{Q}(\underline{k}) = \{$half-trans. surf with conical angles $k_i\pi\}/$cut and paste

$\rightarrow$ Complex orbifold of dimension $d = \dim_{\mathbb{C}} \mathcal{Q}(\underline{k}) = 2g - 2 + \ell(\underline{k})$.

- Strata are locally modeled on $\mathbb{C}^d$ via the period map
- Lebesgue measure on local coordinates is preserved by change of coordinates
- Masur-Veech volume of the stratum:

$$\operatorname{Vol} \mathcal{Q}(\underline{k}) = c_{\underline{k}} \cdot \mu \left(\left\{ (X, q) \in \mathcal{Q}(\underline{k}),\ \int_X |q| \leq 1 \right\} \right).$$

- MV volumes can be evaluated via the count of integer points (square-tiled surfaces)

$$\operatorname{Vol} \mathcal{Q}(\underline{k}) = c'_{\underline{k}} \lim_{N \to \infty} \frac{1}{N^d} \operatorname{Card} \left\{ (X, q) \in \mathcal{Q}^{\mathbb{Z}}(\underline{k}),\ \int_X |q| \leq N \right\}.$$

History

Computation of volumes of strata of *quadratic* differentials

- Eskin-Okounkov $\sim$ '05: algorithms for small dimension
- Athreya-Eskin-Zorich '12: closed formulas for volumes in genus 0
- Delecroix-G-Zograf-Zorich '18, $\mathrm{Vol}(\mathcal{Q}_{g,p})$ as a sum over stable graphs
- Andersen-Borot-Charbonnier-Delecroix-Giacchetto-Lewanski-Wheeler '19: topological recursion for $\mathrm{Vol}(\mathcal{Q}_{g,p})$ (from DGZZ '18)
- Chen-Möller-Sauvaget '19 $\mathrm{Vol}(\mathcal{Q}_{g,p})$ and odd strata as Hodge integrals, and $p \to \infty$
- Aggarwal '19 $\mathrm{Vol}(\mathcal{Q}_{g,p})$ as $g \to \infty$ based on [DGZZ '18].
- Kazarian '19 , Yang-Zagier-Zhang '20: quadratic recursion for $\mathrm{Vol}(\mathcal{Q}_{g,p})$ based on [CMS].
- Sauvaget '21 recursion for odd strata

Formula for the Masur-Veech volumes of odd strata

Theorem (Duryev-G-Yakovlev, 2025)

$$\overline{\mathrm{Vol}}\mathcal{Q}(\underline{k}) := c_d \sum_{\Gamma \in \mathcal{G}_{g,n}^{[\underline{k}]}} \mathcal{Z}\left(\underbrace{\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{v\in V_\Gamma}\frac{N_{g_v,n_v}^{[\underline{k}_v]}(\underline{b}_v)}{2}\prod_{e\in E_\Gamma} b_e}_{P_\Gamma}\right)$$

$$= \mathrm{Vol}\,\mathcal{Q}(\underline{k}) + \sum_{\underline{k}'<\underline{k}}\sum_{\underline{g}} C_{\underline{k}',\underline{g}}\,\mathrm{Vol}\left(\prod \mathcal{H}(2g_i-2)\times\mathcal{Q}(\underline{k}')\right)$$

with $d = \dim_{\mathbb{C}}\mathcal{Q}(\underline{k})$, $\mathcal{G}_{g,n}^{[\underline{k}]}$ is a set of decorated stable graphs, $\mathcal{Z}$ is a linear operator on multivariate polynomials, and $N_{g,n}^{[\underline{k}]}$ are the Kontsevich polynomials.

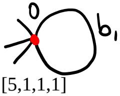
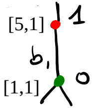
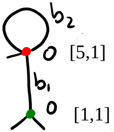
Example: $\mathcal{Q}(5,1,1,1)$

$d = 4, \quad c_d = 2^{d+2}/(d-1)! = 32/3,$

$N_{0,3}^{[5,1]}(b_1, b_2, b_3) = 3 \qquad N_{0,1}^{[1^2]} = 1$

$N_{1,1}^{[5,1]}(b_1) = \frac{b_1^2}{8} \qquad N_{0,2}^{[5,1^3]}(b_1, b_2) = \frac{3}{4}(b_1^2 + b_2^2)$

Example: $\mathcal{Q}(5,1,1,1)$

$\Gamma \in \mathcal{G}_{1,3}^{[5,1^3]}$	P_Γ	$c_d\mathcal{Z}(P_\Gamma)$
	$\frac{1}{2} b_1 \frac{N_{0,2}^{[5,1^3]}(b_1,b_1)}{2}$	$24\zeta(4) = \frac{4\pi^4}{15}$
	$3b_1 \frac{N_{1,1}^{[5,1]}(b_1)}{2} \frac{N_{0,1}^{[1^2]}(b_1)}{2}$	$6\zeta(4) = \frac{\pi^4}{15}$
	$3\frac{1}{2} b_1 b_2 \frac{N_{0,1}^{[5,1]}(b_1,b_2,b_2)}{2} \frac{N_{0,1}^{[1^2]}(b_1)}{2}$	$12\zeta(2)^2 = \frac{\pi^4}{3}$

TOTAL:

$$\overline{\mathrm{Vol}}\mathcal{Q}(5,1^3) = \frac{2\pi^4}{3} = \mathrm{Vol}\mathcal{Q}(5,1^3) + \mathrm{Vol}(\mathcal{H}(2)\times\mathcal{Q}(1^4)) = \frac{5\pi^4}{9} + \frac{\pi^4}{9}$$

Idea of the proof

Formula for the completed volumes

- To compute volumes of strata, count the number of integer points (= square-tiled surfaces) with area $\leq N$ and let $N \to \infty$

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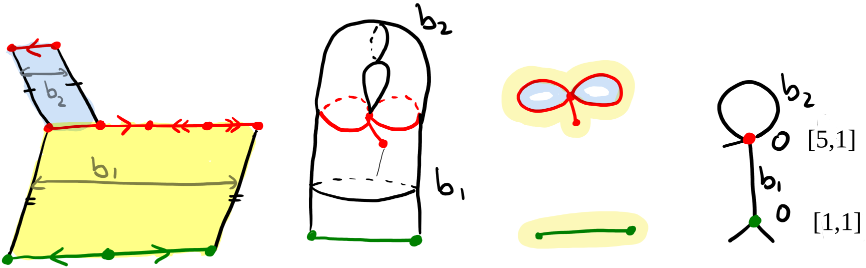
- Square-tiled surfaces always decompose into horizontal cylinders. The decomposition is encoded in a decorated stable graph.
- Horizontal cylinders are glued along ribbon graphs with prescribed odd valencies, the problem is reduced to count the number of integer metrics on such graphs.



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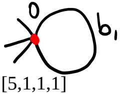
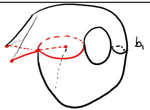
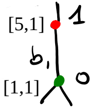
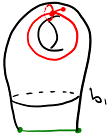
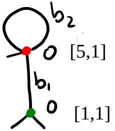
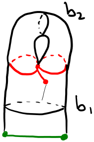
		$\frac{1}{2} b_1 \frac{N_{0,2}^{[5,1^3]}(b_1, b_1)}{2}$	$24\zeta(4) = \frac{4\pi^4}{15}$
		$3 b_1 \frac{N_{1,1}^{[5,1]}(b_1)}{2} \frac{N_{0,1}^{[1^2]}(b_1)}{2}$	$6\zeta(4) = \frac{\pi^4}{15}$
		$3 \frac{1}{2} b_1 b_2 \frac{N_{0,1}^{[5,1]}(b_1, b_2, b_2)}{2} \frac{N_{0,1}^{[1^2]}(b_1)}{2}$	$12\zeta(2)^2 = \frac{\pi^4}{3}$

TOTAL: $\overline{\mathrm{Vol}}\mathcal{Q}(5, 1^3) = \frac{2\pi^4}{3}$. (compare to $\mathrm{Vol}\mathcal{Q}(5, 1^3) = \frac{5\pi^4}{9}$)

Back to the example: $\mathcal{Q}(5, 1^3)$

$$
\begin{array}{ccccc}
\overline{[5]} & = & [5] & + & {\color{red}1} \cdot [2][1] \\
\overline{\mathrm{Vol}}\,\mathcal{Q}(5,1^3) & = & \mathrm{Vol}\,\mathcal{Q}(5,1^3) & + & \mathrm{Vol}\left(\mathcal{H}(2) \times \mathcal{Q}(1^4)\right) \\
\frac{2}{3}\pi^4 & = & \frac{5}{9}\pi^4 & + & \frac{1!1!}{3!}\left(\frac{\pi^2}{3} \cdot 2\pi^2\right)
\end{array}
$$

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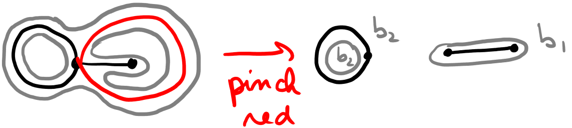
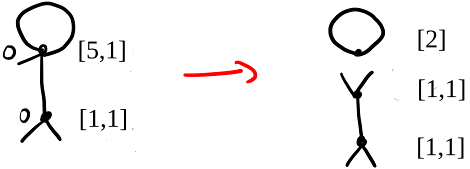
\begin{aligned}
\overline{[5]} &= [5] + {\color{red}1} \cdot [2][1] \\
\overline{\mathrm{Vol}}\,\mathcal{Q}(5,1^3) &= \mathrm{Vol}\,\mathcal{Q}(5,1^3) + \mathrm{Vol}\left(\mathcal{H}(2) \times \mathcal{Q}(1^4)\right) \\
\frac{2}{3}\pi^4 &= \frac{5}{9}\pi^4 + \frac{1!1!}{3!}\left(\frac{\pi^2}{3} \cdot 2\pi^2\right)
\end{aligned}
$$



(local)



(global)

Proof of the formula for the count of metrics

Assume $\ell(\underline{k}) \geq 3$.

- Evaluate the difference between $N^{\underline{k}}_{g,n}(b_1, \ldots, b_n)$ and the counting function for the metrics on ribbon graphs, on walls $\sum b_i = \sum b_j$.
- The only contributing degenerations on these walls come from pinching loops separating "abelian" components.

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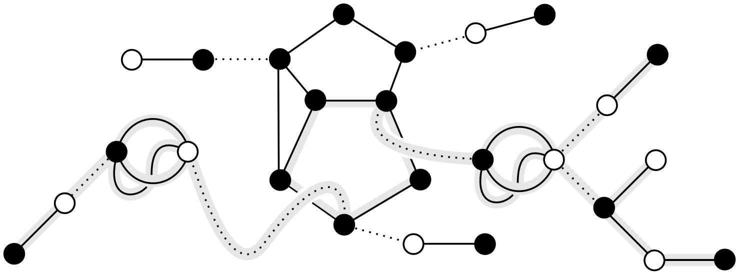
- The only contributing degenerations on these walls come from pinching loops separating "abelian" components.
- Compute the contribution of those degenerated graphs.



Large genus asymptotics

Application to large genus asymptotics

The results [DGZZ] and [Agg] lead to several results in the case of principal strata (singularities of degree 3 only):

- Separating 1-cylinder square-tiled surfaces

Theorem

$$\frac{c(sep)}{c(nonsep)} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g} \quad as\ g \to \infty.$$

- Proportion of 1-cylinder surfaces

Theorem

$$\frac{cyl_1}{\mathrm{Vol}} \sim \sqrt{\frac{\pi}{24g}} \quad as\ g \to \infty.$$

Application to large genus asymptotics

The results [DGZZ] and [Agg] lead to several results in the case of principal strata (singularities of degree 3 only):

- Distribution of number of cylinders

Theorem

It converges in a strong sense to the Poisson distribution of parameter $\lambda_g = \log(6g-6)/2$.

Same convergence as the the number of cycles of random permutations to $\mathrm{Poi}_{\log(n)}$ [Hwang,Nikeghbali-Zeindler].

- Global separation:

Theorem

All singularities of a SQT are located on the same horizontal layer with probability that tends to 1 when g tends to infinity.

Application to large genus asymptotics

Conjecture (work in progress)

All these results hold for odd strata $\mathcal{Q}(\underline{k}, 3^m)$ as $g \to \infty$ (so $m \to \infty$).

Note: 1-cylinder contributions to the "completed" volume correspond exactly to 1-cylinder contributions to the "usual" volume. Contribution of k-cylinders surfaces to the "completed" (versus "usual") volume is more intricate.

Idea: intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{\underline{k}}$ with k_i fixed (except $k_i = 1$) behave asymptotically similarly to classical intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$.