Descent-biased trees and dendrons
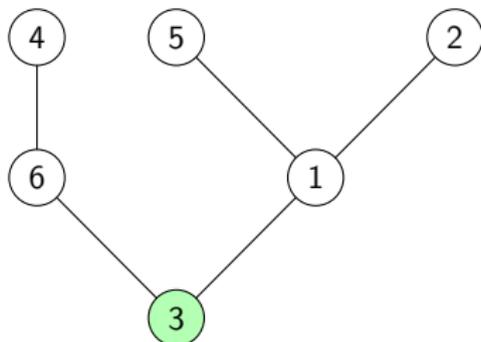## with Victor Dubach (Uppsala) and Stephan Wagner (TU Graz)

Paul Thévenin

Séminaire Flajolet

February 5, 2026

Trees

- Tree : connected acyclic graph.

- One marked vertex: the root. Oriented from the root towards the leaves.

- Vertices are labelled from 1 to n, where n is the number of vertices.

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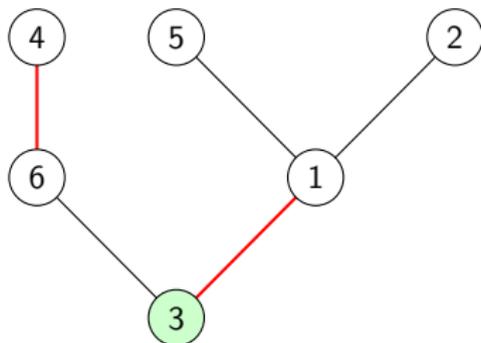
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- An edge $e \in t$ is a *descent* if it goes from a vertex to another vertex with smaller label. $d(t)$ number of descents in a tree t.

Example

$$d(t) = 2$$

Random descent-biased tree

- Fix $q \in [0, 1]$, and let $\mathcal{T}_n^{(q)} \in \mathfrak{T}_n$ be the random tree such that, for all $t \in \mathfrak{T}_n$:

$$\mathbb{P}\left(\mathcal{T}_n^{(q)} = t\right) = \frac{1}{Z_{n,q}} q^{d(t)},$$

where $Z_{n,q} = \sum_{t \in \mathfrak{T}_n} q^{d(t)}$.

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- Interpolates between two well-known models: random recursive trees ($q = 0$) and uniform Cayley trees ($q = 1$).
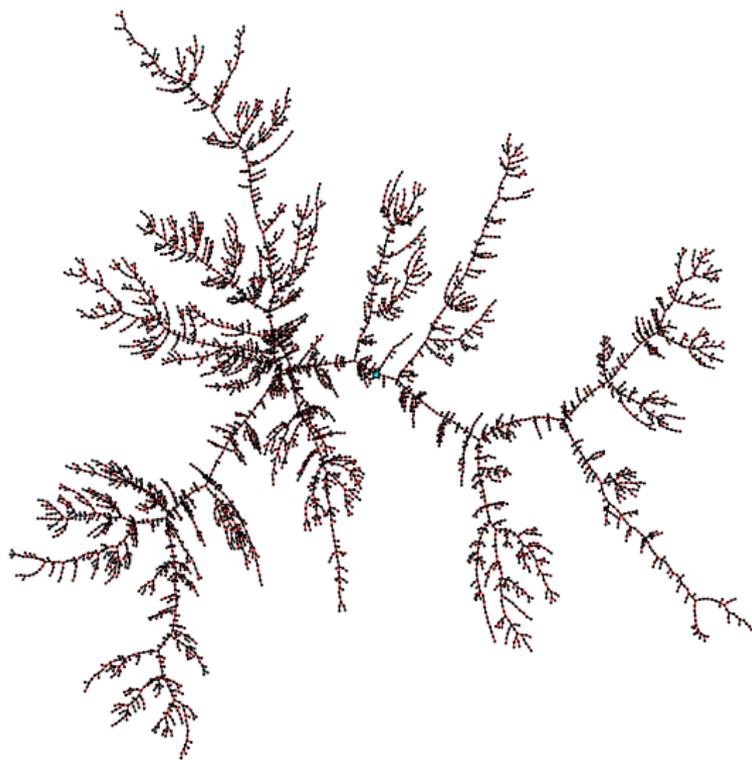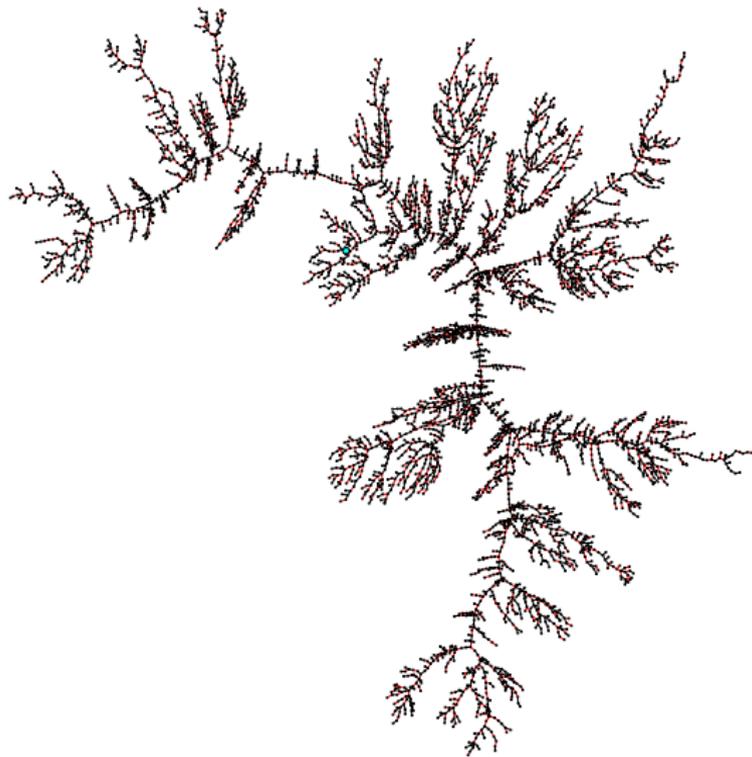
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- Interpolates between two well-known models: random recursive trees ($q = 0$) and uniform Cayley trees ($q = 1$).

- Goal: "study the structure" of $\mathcal{T}_n^{(q)}$ for n large.
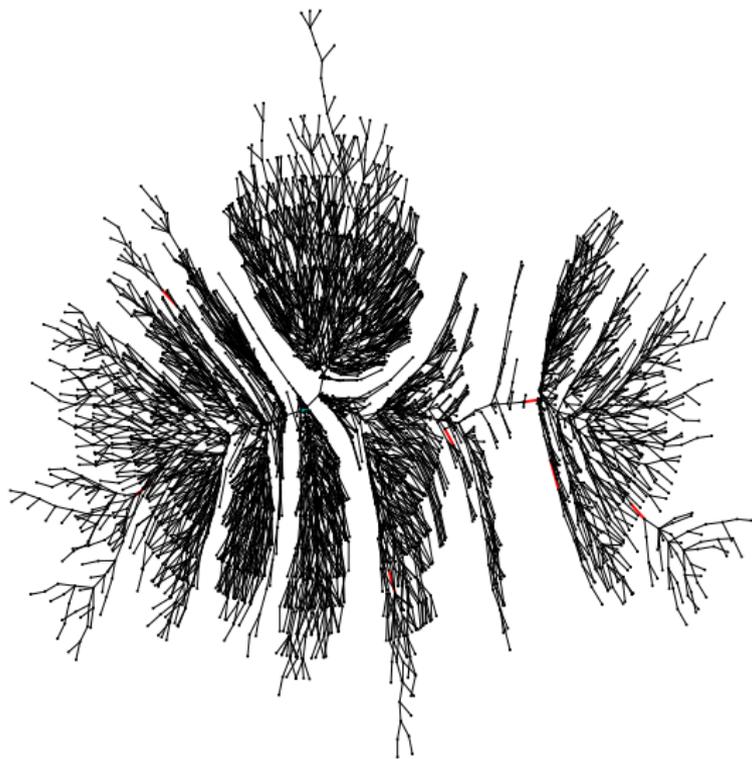
$n = 5000, q = 1$ (uniform Cayley tree)
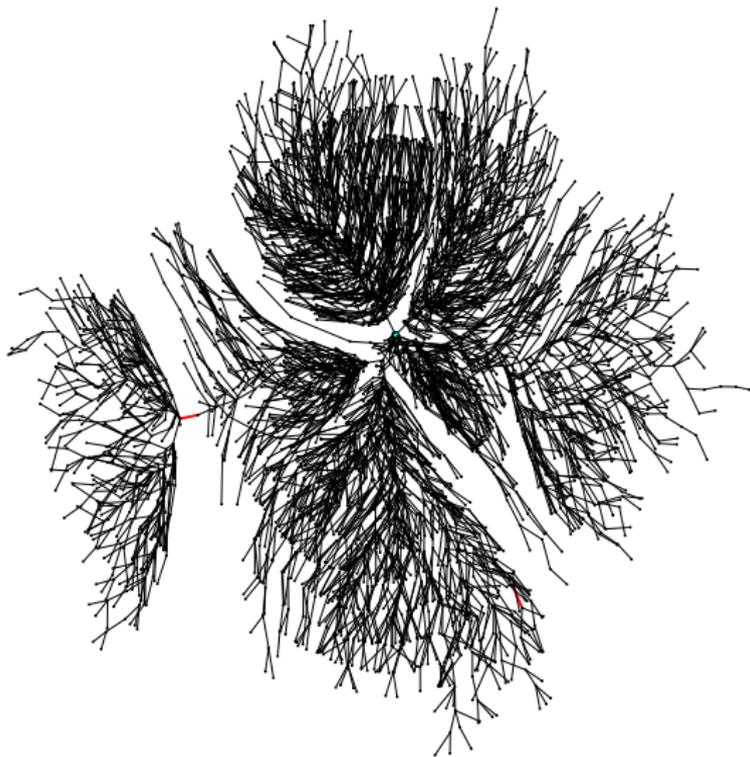
$n = 5000, q = 0.5$

$n = 5000, q = 1/1000$

$n = 5000, q = 1/2500$

$n = 5000, q = 1/10000$

$n = 5000, q = 0$ (random recursive tree)
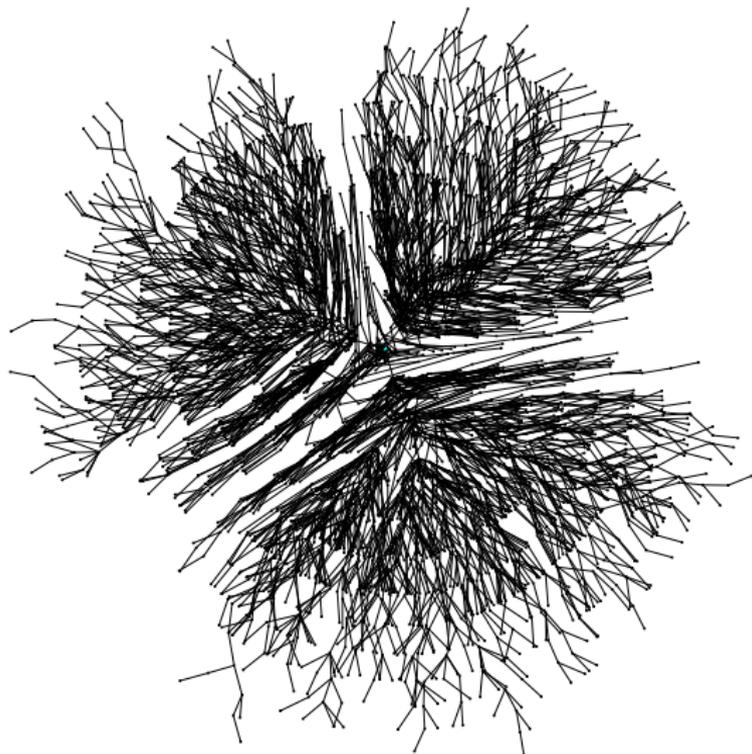
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- *Local* limit of descent-biased permutations and trees [T. & Wagner '23]: what do finite neighbourhoods of the root look like?

- *Scaling* limit of descent-biased trees? [Dubach, T. & Wagner '26+].
  Distances between uniform vertices.

Metric-measure space and convergence

Rooted m.m. space

A rooted metric-measure space (m.m. space) $(\mathcal{X}, \rho, d, \mu)$ is a metric space $(\mathcal{X}, d)$, with a marked point ρ and a probability measure μ.

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We see a finite rooted tree (T, ρ) as a m.m. space $(T, \rho, d_T, Unif_T)$, where

- d_T is the usual graph distance;
- $Unif_T$ the uniform measure on its vertices.

Metric-measure space and weak convergence

Weak convergence

Let $((\mathcal{X}_n, \rho_n, d_n, \mu_n))_{n \in \mathbb{N} \cup \{\infty\}}$ be rooted m.m. spaces, and $(a_n)_{n \geq 1}$ a sequence of positive numbers. Let $(u_i^{(n)})_{1 \leq i \leq p}$ be i.i.d. points in $\mathcal{X}_n$ with distribution μ_n, and $u_0^{(n)} := \rho_n$.

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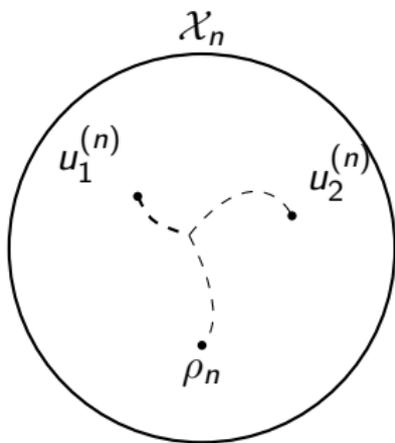
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- We only look at a finite number of points $\rightarrow$ suited for combinatorial methods.

$$a_n \begin{pmatrix} 0 & d(\rho_n, u_n^{(1)}) & d(\rho_n, u_n^{(2)}) \\ d(\rho_n, u_n^{(1)}) & 0 & d(u_n^{(1)}, u_n^{(2)}) \\ d(\rho_n, u_n^{(2)}) & d(u_n^{(1)}, u_n^{(2)}) & 0 \end{pmatrix}$$

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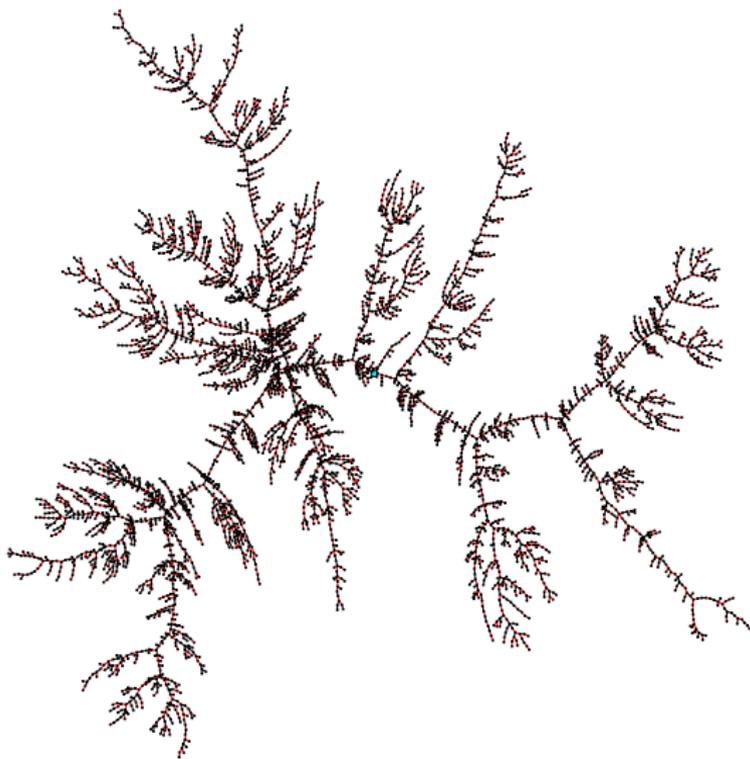
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$$\left(a_n \, d_{\mathcal{T}_n}(u_i^{(n)}, u_j^{(n)}) \right)_{0 \leq i,j \leq p} \overset{(d)}{\underset{n \to \infty}{\to}} \left(d_\infty(u_i^{(\infty)}, u_j^{(\infty)}) \right)_{0 \leq i,j \leq p}$$

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When $q = 1$, $\mathcal{T}_n^{(1)}$ is a uniform Cayley tree with n labelled vertices.

- We can count them : $|\mathfrak{T}_n| = n^{n-1}$ [Borchardt 1860, Cayley 1889].

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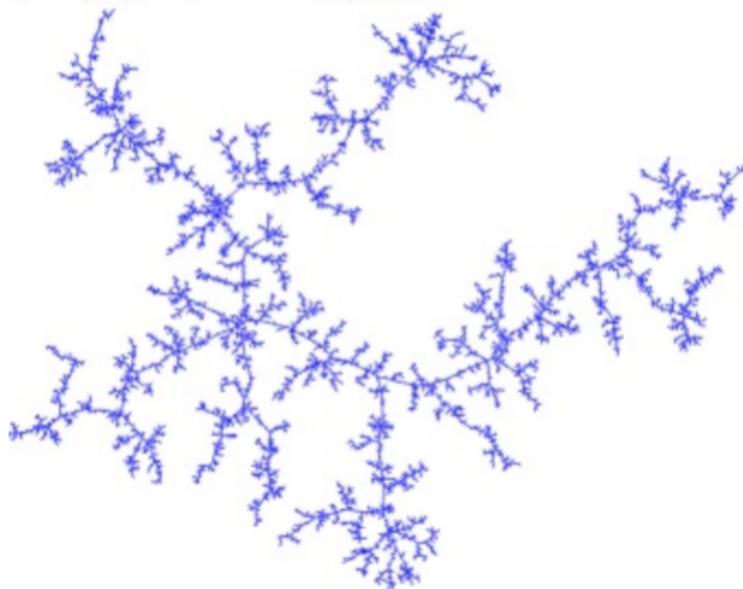When $q = 1$, $\mathcal{T}_n^{(1)}$ is a uniform Cayley tree with n labelled vertices.

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- Scaling limit: Brownian tree $\mathcal{T}_\infty$ [Aldous '91,'93]

The Brownian tree $\mathcal{T}_\infty$

Convergence for $q = 1$

Theorem [Aldous '91 '93], [Le Gall '05]

Let $(\mathcal{T}_n^{(1)}, \rho_n)$ be a uniform Cayley tree with n vertices. Let $(\mathcal{T}_\infty, \rho_\infty, d_\infty, \mu_\infty)$ be Aldous' Brownian tree. Let u_n, v_n be two i.i.d. uniform vertices in $\mathcal{T}_1^{(n)}$, and u, v two i.i.d. points of $\mathcal{T}_\infty$ with law μ_∞. Then, in distribution, as $n \to \infty$:

$$\frac{1}{\sqrt{n}} d_n(u_n, v_n) \xrightarrow{(d)} d_\infty(u, v), \frac{1}{\sqrt{n}} d_n(\rho_n, u_n) \xrightarrow{(d)} d_\infty(\rho_\infty, u).$$
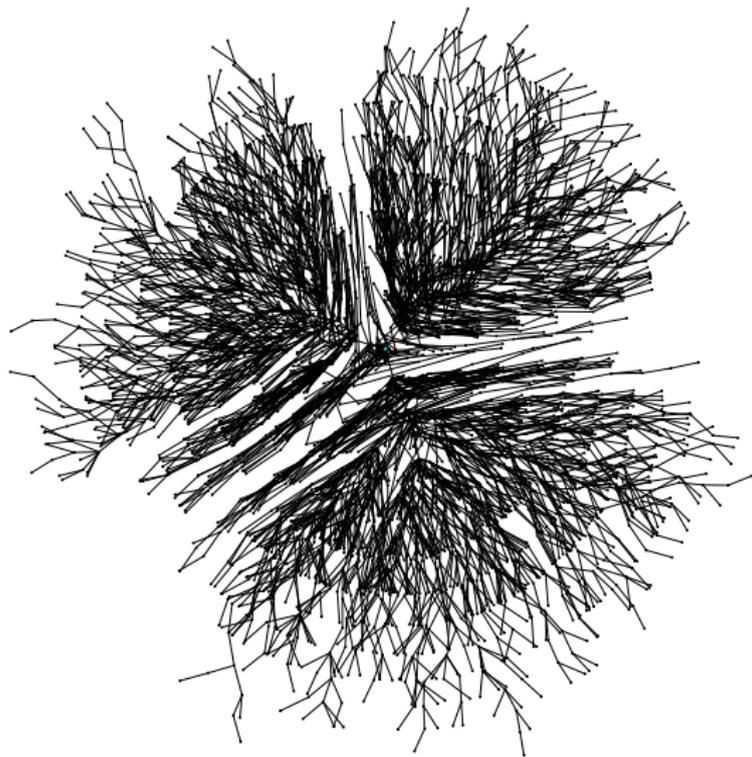
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True for several uniform i.i.d. vertices: for all fixed $p \geq 1$,

$$\frac{1}{\sqrt{n}} (d_n(u_n^{(i)}, u_n^{(j)}))_{1 \leq i,j \leq p} \xrightarrow[n \to \infty]{(d)} (d_\infty(u^{(i)}, u^{(j)}))_{1 \leq i,j \leq p}.$$

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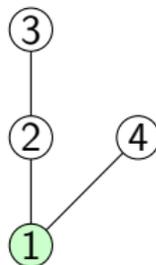When $q = 0$, $\mathcal{T}_n^{(0)}$ is a random recursive tree, zero descent.

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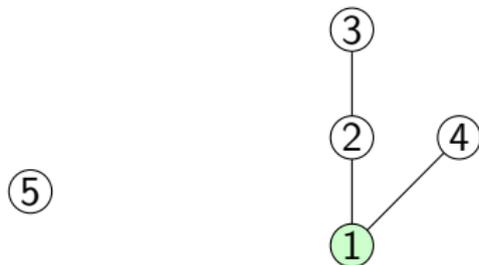When $q = 0$, $\mathcal{T}_n^{(0)}$ is a random recursive tree, zero descent.

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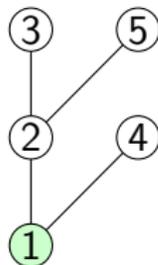
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Some results

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- Height of the tree $\mathcal{T}_n^{(0)} \sim e \log n$ [Pittel '94].

- Height of a typical vertex in $\mathcal{T}_n^{(0)} \sim \log n$.

"Scaling limit" for random recursive trees

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Lemma

Let u_n, v_n two i.i.d. uniform vertices in $\mathcal{T}_n^{(0)}$, and ρ_n its root. Then we have:

$$\frac{1}{\log n} d_n(\rho_n, u_n) \xrightarrow[n\to\infty]{(\mathbb{P})} 1; \frac{1}{\log n} d_n(u_n, v_n) \xrightarrow[n\to\infty]{(\mathbb{P})} 2.$$

"Scaling limit"

- One still has convergence of distance matrices:

$$\frac{1}{\log n} \left(d(u_n^{(i)}, u_n^{(j)}) \right)_{0 \le i,j \le 2} \underset{n \to \infty}{\to} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

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- In order to define a limiting space: dendrons [Elek & Tardos '22].

Dendrons

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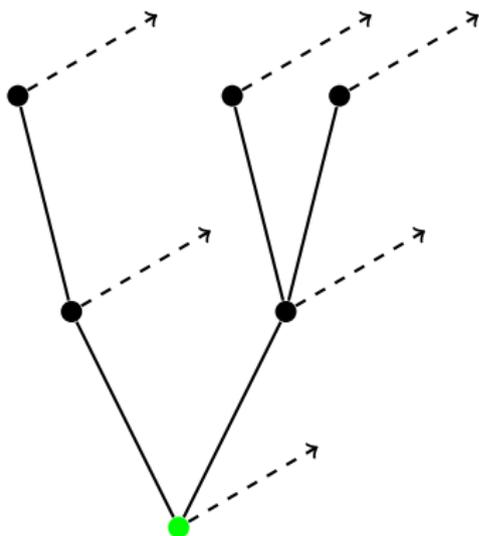
> ### Definition [Elek & Tardos '22]
>
> A *dendron* $D = (T, \rho, d, \nu)$ is a rooted tree (T, ρ, d) along with a probability measure ν on the set $A_D := T \times [0, \infty)$.
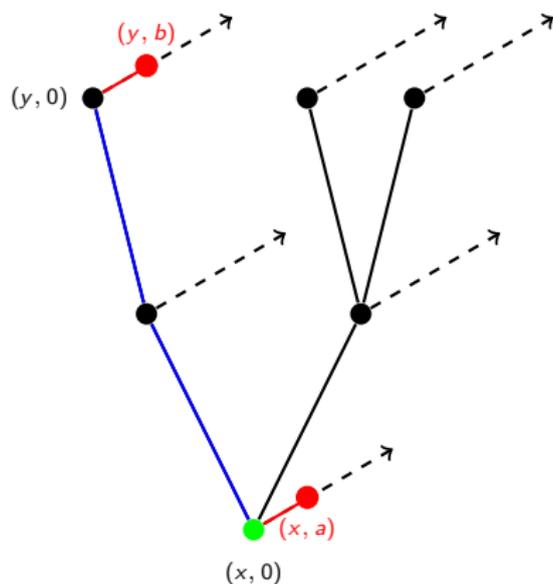> We define a "pseudo-distance" function $d_D : A_D^2 \to \mathbb{R}_+$ by: $\forall x, y \in T, \forall a, b \geq 0$:
>
> $$d_D((x, a), (y, b)) := d(x, y) + a + b.$$
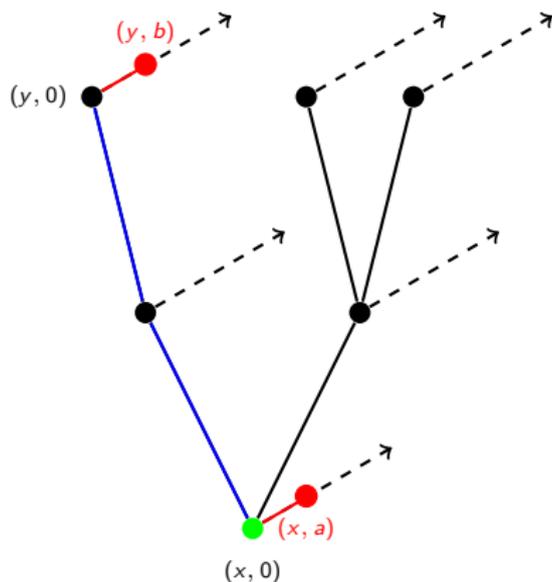
Dendron structure

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$$d_D((x, a), (y, b)) = a + d(x, y) + b.$$

Dendron structure

d_D is not a distance! $d_D((x, a), (x, a)) = a + a$.

Convergence towards dendrons

Definition [Elek & Tardos '22]

A sequence of (random) trees $(\mathcal{T}_n, \rho_n)$ converges towards a dendron (T, ρ, d, ν) if, for all $p \geq 1$:

$$\left(d_{\mathcal{T}_n}(u_i^{(n)}, u_j^{(n)}) \right)_{0 \leq i,j \leq p} \xrightarrow[n \to \infty]{(d)} \left(d_D(u_i, u_j) \right)_{0 \leq i,j \leq p}.$$

Here, $\{u_i^{(n)}, i \geq 1\}$ are i.i.d. $\sim Unif_{\mathcal{T}_n}$, $\{u_i, i \geq 1\}$ are i.i.d. $\sim \nu$, $u_0^{(n)} = \rho_n$ and $u_0 = \rho$.

Limits of trees and dendrons

Theorem [Elek & Tardos '22], [Janson '21]

- Any sequence of (random) trees $(\mathcal{T}_n, \rho_n)_{n \geq 1}$ whose (renormalized) matrices of distances converge in distribution, converges towards a unique dendron.
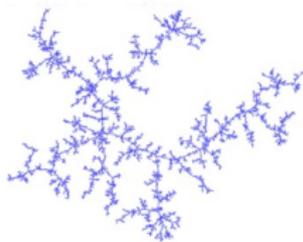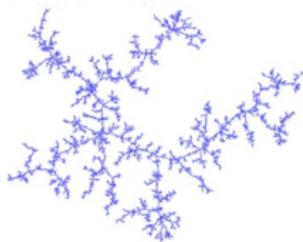
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- Any dendron is the limit of a sequence of (renormalized) trees.
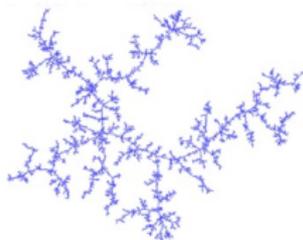
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- Let u, v i.i.d. $\sim \nu$. Then $u = (x, 0), v = (y, 0) \in \mathcal{T}_\infty$, and

$$d_D(u, v) = d_D((x, 0), (y, 0)) = d_\infty(x, y).$$

- $\frac{1}{\sqrt{n}} \mathcal{T}_n^{(1)} \to \mathcal{T}_\infty$.
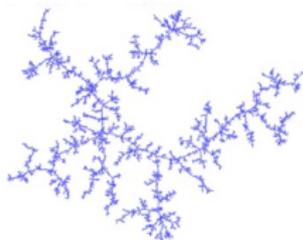
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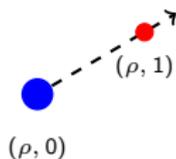$$d_D(u, v) = d_D((x, 0), (y, 0)) = d_\infty(x, y).$$

- $\frac{1}{\sqrt{n}} \mathcal{T}_n^{(1)} \to \mathcal{T}_\infty$.
- All trees are dendrons.

Dendron limit for $\mathcal{T}_n^{(0)}$

$(\rho, 1)$

$(\rho, 0)$

- Underlying tree $(T, d) = \{\rho\}$ (trivial distance).

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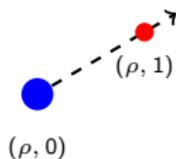
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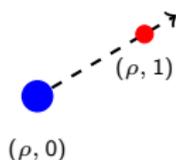
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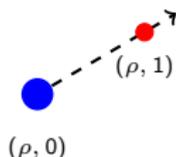
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- Hence, $\mathcal{T}_n^{(0)} \overset{w}{\underset{n \to \infty}{\to}} \Upsilon_{\delta_1}$.
- Notation $\Upsilon_\mu := \{\rho\} \times [0, \infty)$ with measure μ on $[0, \infty)$.

Other examples

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- Split trees, random recursive trees, some trees with logarithmic height, converge to Υ_{δ_1} [Janson '21];

- Uniform attachement trees with freezing converge (in some regime) towards Υ_μ for μ a measure with density [BBRKK '23].

Convergence of descent-biased trees

Théorème [Dubach, T. & Wagner '26+]

Let $(q_n)_{n \geq 1}$ be a sequence of elements in $[0, 1]$. Then we have (in the dendron convergence sense):

- (supercritical) If $q_n = q > 0$, then $\frac{q-1}{\sqrt{q} \log q} \frac{1}{\sqrt{n}} \mathcal{T}_n^{(q)} \to \mathcal{T}_\infty$.

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- (subcritical) If $n \, q_n \to 0$, then $\frac{1}{\log n} \mathcal{T}_n^{(q_n)} \to \Upsilon_{\delta_1}$.

- (critical) If $n \, q_n \to a \in (0, +\infty)$, then $\frac{1}{\log n} \mathcal{T}_n^{(q_n)} \to \mathfrak{D}_a$.

$(\mathfrak{D}_a)_{a>0}$: family of non-trivial dendrons.

The "supercritical" phase $q_n = q$ fixed

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- Main tool: decomposition of the tree $\mathcal{T}_n^{(q)}$.

Tree decomposition

$$A(x, q) := \sum_t \frac{x^{|t|}}{|t|!} q^{d(t)}$$

Either the vertex labelled 1 is the root of the tree, or it is the endpoint of a descent.
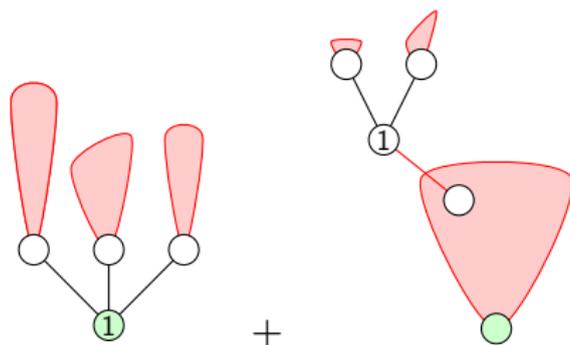
Singularity analysis

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$$\frac{\partial}{\partial x} A(x, q) = e^{A(x,q)} + qx e^{A(x,q)} \frac{\partial}{\partial x} A(x, q)$$

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- Singularity analysis around the unique singularity of $A(x,q)$ provides

$$[x^n] A(x,q) \sim \frac{1}{\sqrt{2\pi q}} n^{-3/2} \left(q^{q/(1-q)} \right)^{-n}$$

Height of a uniform vertex (1)

In order to get the distribution of the height of a uniform vertex, we compute its moments. Let

$$G(x, q, y) := \sum_t \sum_{w \in t} \frac{x^{|t|}}{|t|!} q^{d(t)} y^{h(w)},$$

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We have

$$\mathbb{E}[h(u_n)] = \frac{\frac{1}{n}[x^n]\frac{\partial}{\partial y} G(x, q, y)_{|y=1}}{[x^n]A(x, q)},$$

where u_n is uniform in $\mathcal{T}_n^{(q)}$.

Height of a uniform vertex (2)

Singularity analysis provides

$$\mathbb{E}\left[\frac{q-1}{\sqrt{q}\log q}\frac{h(u_n)}{\sqrt{n}}\right] \underset{n\to\infty}{\to} C,$$

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Heavier computations allow us to extend it to distances between p-tuples of i.i.d. uniform vertices.

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Theorem [Dubach, T. & Wagner '26+]

Fix $a > 0$, and $(q_n) \in (0, 1)$ such that $nq_n \to a$. Then, in the dendron convergence sense, we have:

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- The construction is a continuous version of the "Foata-Fuchs bijection" between lists of integers and finite trees with a prescribed sequence of degrees.

Structure of $\mathfrak{D}_a$

- In the regime $nq_n \to a$, one can still compute asymptotics of coefficients of $A(x, q_n)$:

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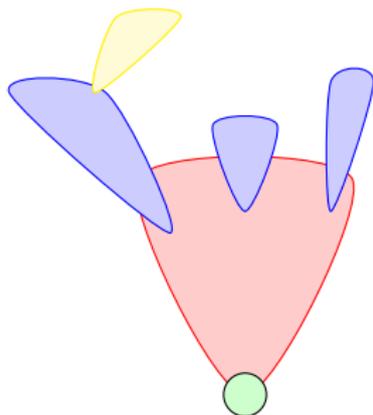- In particular, one can compute the limiting distribution for the height of a uniform vertex

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- Hence, almost all vertices in $\mathcal{T}_n^{(a/n)}$ have a height which is an integer multiple of $\log n$. Structure of the tree?
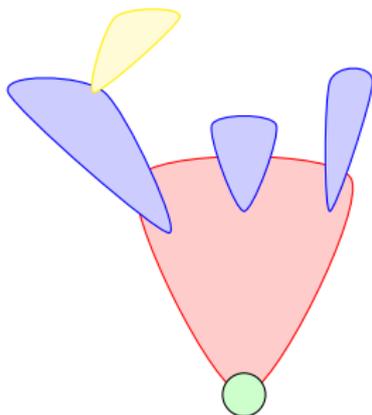
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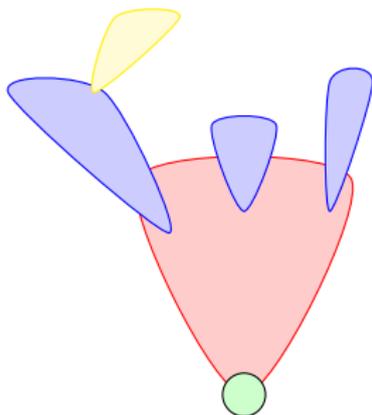
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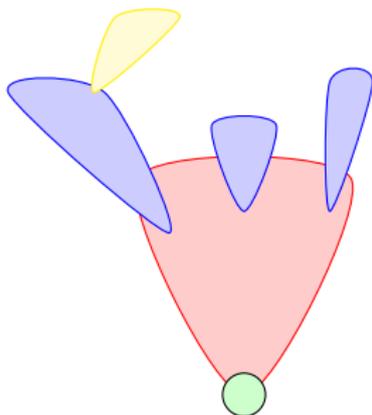
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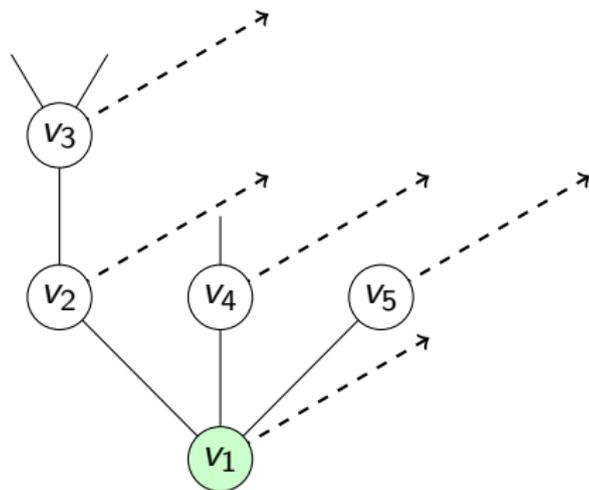

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- Almost all vertices in the yellow part have height $3 \log n$.

Structure of the tree: $q_n = 1/n, n \gg 1$

Components without descents

Structure of the dendron
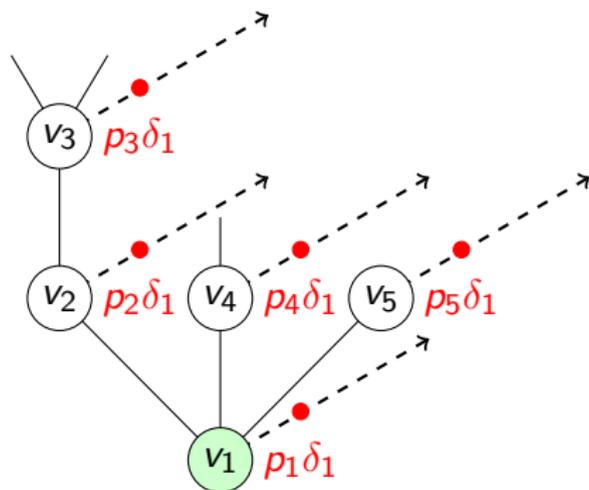
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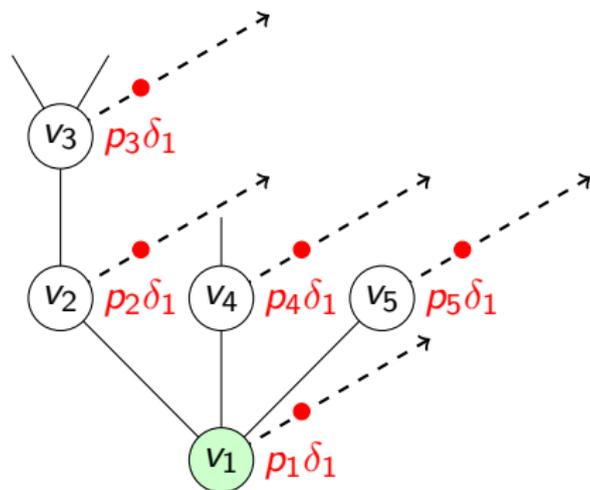
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The tree structure and the weights $(p_i)_{i \geq 1}$ are random.

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- We obtain the precise law of the limiting dendron, and a recursive construction.
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- Dendrons allow us to construct limiting spaces for trees with degenerate behaviours, in which distances between uniform vertices converge in distribution.
- Weak notion of convergence ; proving the convergence "boils down to" studying distances between finitely many points.

THANKS
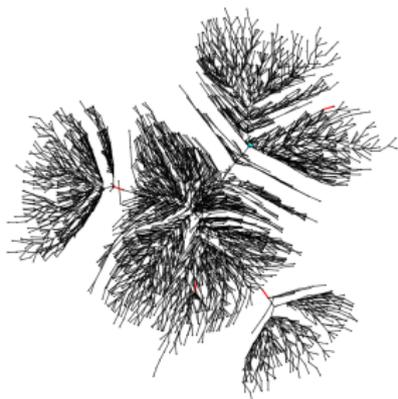
The limiting dendron $\mathfrak{D}_a$

Construction of the base tree, which is an infinite tree with vertices $(v_i)_{i \geq 1}$.

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- Edges have length one.

An example

$$[3, 5, 2, 3, 1, 2, 5, 7, \ldots]$$

③

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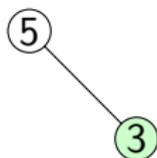
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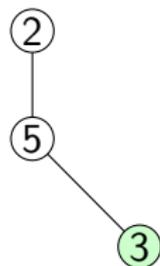
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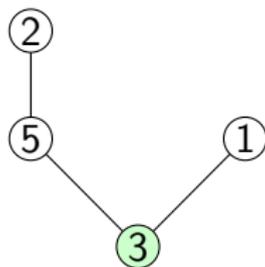
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Composante racine

- L'arbre à descentes $\mathcal{T}_n^{(q_n)}$ est un ensemble d'arbres récursifs greffés les uns sur les autres.
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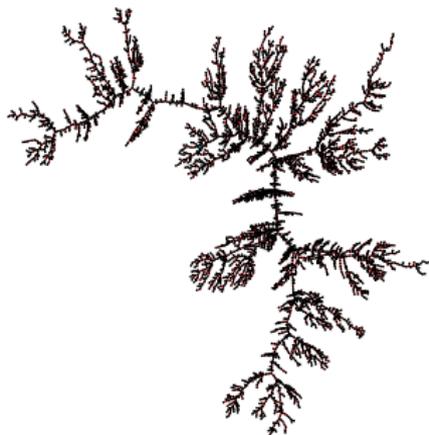
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Composante racine

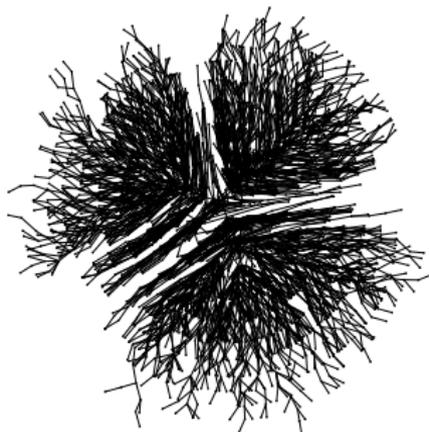- L'arbre à descentes $\mathcal{T}_n^{(q_n)}$ est un ensemble d'arbres récursifs greffés les uns sur les autres.
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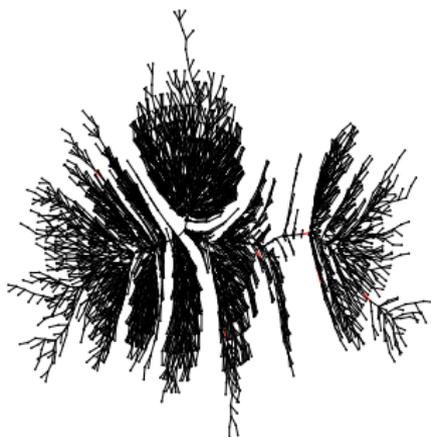
Pour $nq_n \to a \in (0, \infty)$, la composante racine a taille $\Theta(n)$.

Heuristique

- Dans le cas $nq_n \to a$, on peut voir un arbre à descentes comme un ensemble d'arbres récursifs (=composantes) de tailles $\Theta(n)$ greffés les uns sur les autres, selon une structure arborescente.

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- Soient u_n, v_n uniformes dans $\mathcal{T}_n^{(q_n)}$.
  - S'ils sont dans la même composante, $d_n(u_n, v_n) \sim 2\ln(n)$;
  - Sinon, le chemin entre u_n et v_n passe par k composantes, et $d_n(u_n, v_n) \approx \ln(n) + k\ln(n) + \ln(n) = (k+2)\ln(n)$.

Merci !
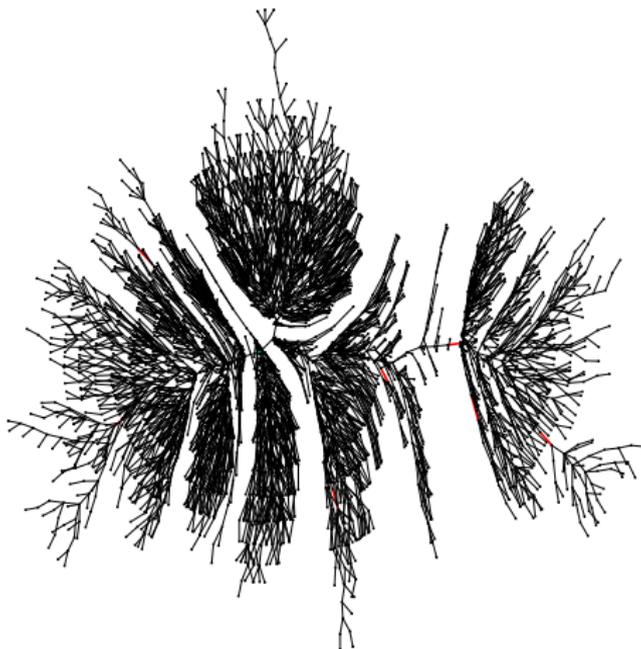
Etude de la composante racine

Reste à estimer les tailles des composantes.

Fonction génératrice des arbres à descentes : $A(x, q) = \sum_t \frac{1}{|t|!} x^{|t|} q^{d(t)}$

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On cherche à estimer les moments de $\frac{c(t)}{n}$.

$$\boxed{\mathbb{E}\left[e^{y \frac{c(\mathcal{T}_n^{(qn)})}{n}}\right] = \frac{[x^n] G(x, \frac{a}{n}, e^{y/n})}{[x^n] A(x, \frac{a}{n})}}$$

La fonction $A(x, q)$

- Décomposition récursive de l'arbre $\mathcal{T}_n^{(q)}$, selon la position de l'étiquette $\textcircled{1}$ $\Rightarrow$ Equation implicite vérifiée par $A(x, q)$:

$$x = \frac{e^{-qA(x,q)} - e^{-A(x,q)}}{1-q}$$

- Par méthode du point col [Flajolet & Sedgewick '09], on obtient

$$[x^n]A(x, \frac{a}{n}) \underset{n \to \infty}{\sim} \frac{n^{a-1}}{e^a \Gamma(a+1)}.$$

La fonction $G(x, q, s)$

On peut exprimer G en fonction de A. Soit T un arbre, $C(T)$ sa composante racine.

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- En remplaçant, on obtient $\boxed{G(x, q, s) = -\log\left(1 - s\left(1 - e^{-A(x,q)}\right)\right)}$

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$$[x^n]G(x, a/n, e^{y/n}) \sim n^{a-1}e^{-a}\sum_{k\geq 0}\frac{y^k}{\Gamma(a+k+1)}.$$

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- Distribution discrète ! $\frac{c(\mathcal{T}_n^{(a/n)})}{n} \sim Beta(1, a)$ densité $a(1-x)^{a-1}$.

Construction de $\mathcal{T}_*^{(q)}$

- $\mathcal{T}_*^{(q)}$ est un arbre "Kesten-like" : une branche infinie, des arbres attachés dessus

- Les arbres attachés ne sont ni indépendants ni identiquement distribués. Plus touffu, petites étiquettes proches de la racine.

La lignée ancestrale de n dans $\mathcal{T}_n^{(q)}$

- Idée de la construction : regarder la lignée ancestrale de n. Soit h_n la hauteur de n.

- Soient $a_0, a_1, \ldots, a_{h_n} =: n$ les étiquettes le long.
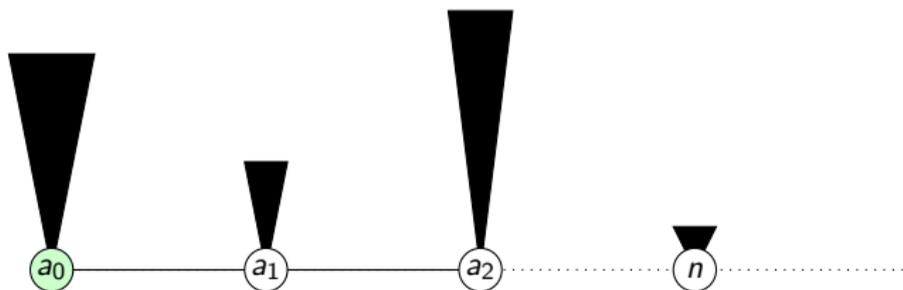
Théorème

> **Lemea [T., Wagner '23+]**
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> Conditionnellement à h_n, $(a_0, \ldots, a_{h_n-1})$ est distribuée comme une permutation q-biaisée de taille h_n.

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- Idée : si on se donne les arbres attachés, les intervertir a le même effet sur le nombre de descentes de $\mathcal{T}_n$ qu'une permutation biaisée.

- Forcément, $a_{h_{n-1}} < n$.

Algorithme

- Cela nous donne un algorithme pour construire la lignée ancestrale de n conditionnellement à h_n.

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- Etape 2 : trier les racines (sauf n) selon une permutation q-biaisée de leurs étiquettes, et on les relie pour former la lignée ancestrale de n.

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Forêt q-biaisée

Lemme [T., Wagner '23+]

Soit F une forêt. Soit N_k le nombre d'arbres de taille k dans F. Alors, pour tout $k \geq 1$, il existe $\pi_k > 0$ tel que

$$\frac{N_k}{h_n} \xrightarrow[n, h_n \to \infty]{(\mathbb{P})} \pi_k.$$

De plus, $\sum_{k \geq 1} \pi_k = 1$.

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- Preuve : fonctions génératrices.

- On a de plus $h_n \gg n^{1/4}$ avec grande probabilité. Cette lignée ancestrale "converge localement" vers une branche infinie.

- En particulier, pour tout k, avec probabilité positive, l'arbre attaché à la racine de $\mathcal{T}_n$ a taille $k < \infty$.

- Finalement, conditionnellement à k, cet arbre est un arbre q-biaisé de taille k.

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- Distribution de $B_r\left(\mathcal{T}_*^{(q)}\right)$: laide, mais explicite.