A q-deformation for enriched P-partitions.

Ekaterina A. Vassilieva

Laboratoire d'Informatique de l'Ecole Polytechnique, 91128 Palaiseau Cedex, France

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Permutations, Young tableaux and symmetric functions

Permutations, descent and peak sets

- $S_n \rightarrow$ symmetric group over n elements.
- $\mathrm{Des}(\pi)$ and $\mathrm{Peak}(\pi) \rightarrow$ **descent set** and **peak set** of $\pi \in S_n$.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 7 & 3 & 1 & 8 & 4 & 5 \end{pmatrix}$$

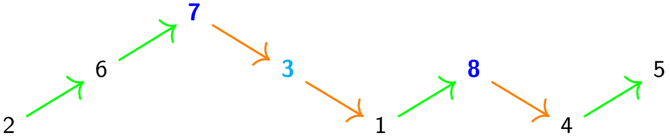


Figure: Ex. $\pi \in S_8$ such that $\mathrm{Des}(\pi) = \{3, 4, 6\}$ and $\mathrm{Peak}(\pi) = \{3, 6\}$.

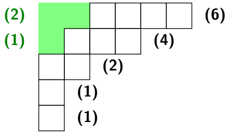
- $\mathrm{Des}(\pi) = \{1 \leq i \leq n-1 | \pi(i) > \pi(i+1)\}$.
- $\mathrm{Peak}(\pi) = \{2 \leq i \leq n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\} \subseteq Des(\pi)$.

Partitions and Young tableaux

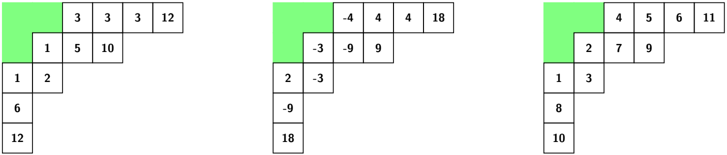
- $\lambda \vdash n \to \lambda$ **integer partition** of n. Ex $(6,4,2,1,1) \vdash 14$ as

$$6+4+2+1+1=14.$$

- $\lambda/\mu \to$ **skew partition**. Ex $(6,4,2,1,1)/(2,1)$
- Skew partitions are represented as **Young diagram**



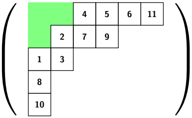
- **Semistandard**, **marked** and **standard** Young tableaux of shape λ/μ are labelling of the boxes.



Partitions and Young tableaux

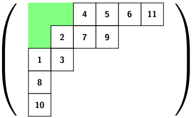
- The **descent and peak sets** of a standard Young tableau T
- $\mathrm{Des}(T) = \{1 \leq i \leq n-1 \mid i$ is in a strict. higher row than $i+1\}$.
- $\mathrm{Peak}(T) = \{2 \leq i \leq n-1 \mid i \in \mathrm{Des}(T)$ and $i-1 \notin \mathrm{Des}(T)\}$.
- For marked tableaux, $\mathrm{neg}(T)$ is the number of negative entries.

Des

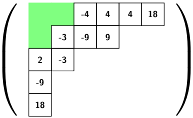


$= \{2, 6, 7, 9\}$

Peak



$= \{2, 6, 9\}$, neg



$= 5$

RS(K) correspondence

- Robinson-Schensted (RS) correspondence $\rightarrow$ bijection between permutations and pairs of standard tableaux of same shape.
- Example $\pi = 413265$

$$\emptyset \xrightarrow{4} \begin{array}{|c|}\hline 4 \\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array} \xrightarrow{3} \dots \xrightarrow{6} \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \xrightarrow{5} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = P_\pi$$

$$\emptyset \xrightarrow{1} \begin{array}{|c|}\hline 1 \\ \hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \xrightarrow{3} \dots \xrightarrow{5} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \xrightarrow{6} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = Q_\pi$$

- If π corresponds to (P_π, Q_π), then π^{-1} corresponds to (Q_π, P_π).
- $Des(\pi) = Des(Q_\pi)$ and $Des(\pi^{-1}) = Des(P_\pi)$.
- RSK correspondence $\rightarrow$ bijection between integer matrices and pairs of semistandard tableaux.

Symmetric functions

- $X = \{x_1, x_2, \dots\} \rightarrow$ alphabet on commutative indeterminates.
- $\Lambda[X] \rightarrow$ **the ring of symmetric functions** on X i.e. the power series f such that for any permutation σ.

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots)$$

- Example: elementary symmetric functions

$$\begin{aligned} e_1 &= x_1 + x_2 + \dots \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \dots \\ e_3 &= x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4 \dots \\ &\dots \end{aligned}$$

- The bases of $\Lambda[X]$ are indexed by integer partitions.
- For $\lambda = (\lambda_1, \lambda_2 \dots)$, the monomial m_λ and power sum p_λ:

$$m_\lambda(X) = \sum_{\alpha \sim \lambda} X^\alpha \quad \text{(summing over rearrangements } \alpha \text{ of } \lambda)$$

$$p_\lambda(X) = p_{\lambda_1}(X) p_{\lambda_2}(X) \dots$$

$$p_k(X) = \sum_i x_i^k \quad \text{(for integer } k)$$

Schur symmetric functions

Fix integer k. Let h_k be the **complete homogeneous** symmetric function of order k:

$$h_k = \sum_{1 \leq i_1 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

Definition (Schur symmetric functions)

The ***Schur symmetric function*** *indexed by skew integer partition* λ/μ *(Jacobi–Trudi definition):*

$$s_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right)_{i,j}.$$

Schur symmetric functions

- The Schur symmetric function indexed by λ/μ is the **generating function for semistandard Young tableaux** of shape λ/μ.

$$s_{\lambda/\mu} = \sum_{T \in SSYT(\lambda/\mu)} X^T$$

$T =$

		3	3	3	12
	1	5	10		
1	2				
6					
12					

$$\Rightarrow \quad X^T = x_1^2 x_2 x_3^3 x_5 x_6 x_{10} x_{12}^2$$

- **Cauchy identity**: $\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j}(1 - x_i y_j)^{-1}$.

Hall littlewood symmetric functions

- Fix $t \in \mathbb{C}$. Define $q_n(X;t) \in \Lambda[X]$ by $q_0(X;t) = 1$ and

$$q_n(X;t) = (1-t)\sum_i x_i^n \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \qquad \text{for all } n > 0,$$

- The $(q_n(X;t))_n$ generate a subalgebra $\Lambda_t[X]$ of $\Lambda[X]$.

Definition (Hall-Littlewood S-sym. functions)

*The **Hall-Littlewood S-symmetric function** indexed by skew integer partition λ/μ*

$$S_{\lambda/\mu}(X;t) := \det\left(q_{\lambda_i - \mu_j - i + j}(X;t)\right)_{i,j}.$$

As such, $S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X)$ and $S_{\lambda/\mu}(X;-1)$ is a variant of Schur's Q-function.

Definition (θ_t homomorphism)

Define the $\mathbb{C}$-algebra morphism $\theta_t : \Lambda[X] \longrightarrow \Lambda_t[X]$ by $\theta_t(h_n)(X) = q_n(X;t)$. As a consequence, $\theta_t(s_{\lambda/\mu})(X) = S_{\lambda/\mu}(X;t)$.

Posets, enriched P-partitions and quasisymmetric functions

Labelled posets

- We look at **partially ordered sets** with labelled vertices.
- This is the main building block to introduce **the ring of quasisymmetric functions**, a significant generalization of symmetric functions, the focus of this talk.
- Basically we have a list of vertices labelled with $1, 2, \ldots, n$ and draw some arrows between some of the vertices.

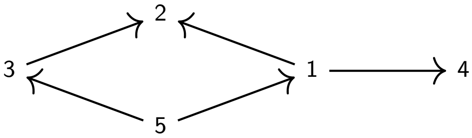


Figure: A 5-vertex labelled poset. Arrows show the covering relations.

Definition (Labelled posets)

Let $[n] = \{1, 2, \ldots, n\}$. A labelled poset $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.

Enriched P-partitions

- Then consider some *additional* labelling of the vertices similarly as semistandard Young tableaux.
- In the basic version, any positive integer can be used for this additional labelling.
- These additional labels are increasing along the arrows. They strictly increase wherever the initial labels are decreasing (!).
- This additional labelling is named a **P-partition**.

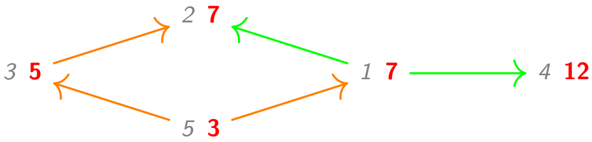


Figure: A P-partition (in red). Arrows indicate the increasing paths. Gray, initial labels define the strictly increasing situations (orange arrows).

Enriched P-partitions

- With the *enriched* version of P-partitions, negative labels are considered with the total order:

$$-1 < 1 < -2 < 2 < \cdots < -i < i < \dots$$

- "Negative equality" is now possible between decreasing initial labels

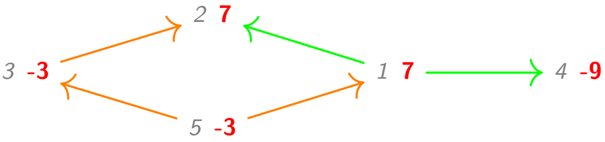


Figure: An enriched P-partition (in red). Arrows indicate the increasing paths. Equal negative red labels are allowed at the extremities of an orange arrow. Equal positive labels are allowed at the extremities of a green arrow.

Enriched P-partitions (formal definition)

Definition (P-partition)

Let $\mathbb{P} = \{1, 2, 3, \dots\}$ and let $P = ([n], <_P)$ be a labelled poset. A P-partition is a map $f : [n] \longrightarrow \mathbb{P}$ that satisfies the two following conditions:

(i) *If $i <_P j$, then $f(i) \leq f(j)$.*

(ii) *If $i <_P j$ and $i > j$, then $f(i) < f(j)$.*

The relations $<$ and $>$ stand for the classical order on $\mathbb{P}$. Let $\mathcal{L}_{\mathbb{P}}(P)$ denote the set of P-partitions.

Definition (Enriched P-partition)

Let $\mathbb{P}^{\pm} = -\mathbb{P} \cup \mathbb{P}$ be the set of positive and negative integers totally ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \dots$. Given a labelled poset $P = ([n], <_P)$, an enriched P-partition is a map $f : [n] \longrightarrow \mathbb{P}^{\pm}$ that satisfies the two following conditions:

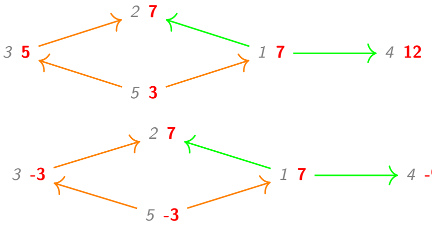
(i) *If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.*

(ii) *If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.*

Further, let $\mathcal{L}_{\mathbb{P}^{\pm}}(P)$ be the set of enriched P-partitions.

Generating functions for labelled posets

- As for Schur functions and semistandard Young tableaux we consider a generating function for labelled posets.
- One red labelling of the poset $\to$ one monomial.
- The indices of the indeterminates are the red labels.
- In the enriched case, negative labels are replaced by their absolute value to determinate the monomial.



$\to x_3 x_5 x_7^2 x_{12}$

$\to x_3^2 x_7^2 x_9$

Generating functions

- Sum over all possible red labelling of a given poset to get the generating function.

Definition

Denote indeterminates $x = \{x_1, x_2, x_3, \ldots\}$*, the ring* $\mathbf{k}[[x]]$ *of formal power series on* x *where* $\mathbf{k}$ *is a commutative ring, and let* $\mathcal{Z} \in \{\mathbb{P}, \mathbb{P}^{\pm}\}$*. Given a labelled poset* $([n], <_P)$*, define its generating function* $\Gamma_{\mathcal{Z}}([n], <_P) \in \mathbf{k}[[X]]$ *by*

$$\Gamma_{\mathcal{Z}}([n], <_P) = \sum_{f \in \mathcal{L}_{\mathcal{Z}}([n], <_P)} \prod_{1 \leq i \leq n} x_{|f(i)|},$$

where $|f(i)| = -f(i)$ *(resp.* $= f(i)$*) for* $f(i) \in -\mathbb{P}$ *(resp.* $\mathbb{P}$*).*

Fundamental and peak quasisymmetric functions

- The case of chain posets (trivially 1-to-1 with permutations) is of crucial interest!

Definition

Given a permutation $\pi = \pi_1 \dots \pi_n$ of S_n, we let $P_\pi = ([n], <_\pi)$ be the labelled poset where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ iff $i < j$.

$$\pi_1 \longrightarrow \pi_2 \longrightarrow \cdots\cdots\cdots \longrightarrow \pi_n$$

Figure: The labelled poset P_π.

Definition

$L_\pi := \Gamma_{\mathbb{P}}([n], <_\pi)$ and $K_\pi := \Gamma_{\mathbb{P}^\pm}([n], <_\pi)$ are Gessel's fundamental (Stembridge's peak) quasisymmetric functions indexed by $\pi \in S_n$.

Fundamental and peak quasisymmetric functions

- L_π and K_π belong to the ring QSym of *quasisymmetric functions*, i.e. for $i_1 < \cdots < i_p$, $x_1^{k_1} \cdots x_p^{k_p}$ and $x_{i_1}^{k_1} \cdots x_{i_p}^{k_p}$ have the same coefficient.
- They are strongly related to our permutations statistics.

Proposition (I. Gessel, J. Stembridge)

$$L_\pi = \sum_{\substack{i_1 \leq \cdots \leq i_n; \\ j \in \mathrm{Des}(\pi) \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$K_\pi = \sum_{\substack{i_1 \leq \cdots \leq i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} 2^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

L_π (K_π) depends only on n and $\mathrm{Des}(\pi)$ ($\mathrm{Peak}(\pi)$). Write $L_{n,\mathrm{Des}(\pi)}$ ($K_{n,\mathrm{Peak}(\pi)}$) instead of L_π (K_π). $(L_{n,I})_{n \geq 0, I}$ and $(K_{n,I})_{n \geq 0, I}$ are the resp. bases of QSym and the algebra of peaks, a subalgebra of QSym.

Link to structure constants of descent and peak algebra

Quasisymmetric functions are related to structure constants.

Definition

For $I \subseteq [n-1]$, let D_I be the formal sum of all permutations of descent set I and d_{IJ}^G is the number of factorisations of a permutation of descent set G into a product of two permutations of respective descent sets I and J

$$D_I D_J = \sum_{G \subseteq [n-1]} d_{IJ}^G D_G$$

Similarly let Π_I be the formal sum of all permutations of peak set I and p_{IJ}^G be the number of factorisations of a permutation of peak set G into a product of two permutations of respective peak sets I and J

$$\Pi_I \Pi_J = \sum_{G \subseteq [n-1],\ \text{peak lacunar}} p_{IJ}^G \Pi_G$$

Link to structure constants of descent and peak algebra

Theorem (I. Gessel, T.K. Petersen)

Let $X = \{x_1, x_2, \dots\}$ *and* $Y = \{y_1, y_2, \dots\}$ *be two alphabets of intedeterminates,* $G \subseteq [n-1]$ *and* H *a peak lacunar subset of* $[n-1]$ *then*

$$L_G(XY) = \sum_{I,J \subseteq [n-1]} d_{IJ}^G L_I(X) L_J(Y)$$

$$K_H(XY) = \sum_{I,J \subseteq [n-1],\ \textit{peak lacunar}} p_{IJ}^H K_I(X) K_J(Y)$$

q-deformed generating functions for enriched P-partitions

q-deformed generating functions

- The generating function

$$\Gamma_{\mathcal{Z}}([n], <_P) = \sum_{f \in \mathcal{L}_{\mathcal{Z}}([n], <_P)} \prod_{1 \leq i \leq n} x_{|f(i)|}$$

does not depend on the sign of the enriched P-partitions f.

- If ω is the map that sends the element i of a labelled poset $([n], <_P)$ and an enriched P-partition f to the contributing monomial in Γ, we have indeed

$$\omega(i, f) = x_{|f(i)|}.$$

$\Rightarrow$ **We break this assumption and add a parameter $\mathbf{q}$:**

$$\omega(i, f, q) = \begin{cases} x_{f(i)} \text{ if } f(i) \in \mathbb{P} \\ \mathbf{q} x_{-f(i)} \text{ if } f(i) \in -\mathbb{P}. \end{cases}$$

q-deformed generating functions

Definition (q-generating series for labelled posets)

Let $q \in \mathbf{k}$ (the base ring of the power series). The q-generating function for enriched P-partitions on the poset $([n], <_P)$ is

$$\Gamma_q([n], <_P) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{i=1}^{n} \omega(i, f, q) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{i=1}^{n} q^{[f(i)<0]} x_{|f(i)|},$$

where $[f(i) < 0] = 1$ if $f(i) < 0$ and 0 otherwise. This definition covers the case of Gessel ($q = 0$) with no negative numbers allowed and the one of Stembridge ($q = 1$) where the sign of f is ignored in the generating function.

q-fundamental quasisymmetric functions

Define the *q-fundammental quasisymmetric function*

Definition (q-fundammental quasisymmetric function)

Let $q \in \mathbf{k}$ and $\pi \in S_n$. The q-fundammental quasisymmetric function indexed by π and parametrised by q is

$$L_\pi^q = \Gamma_q([n], <_\pi).$$

This proposition follows.

Proposition

Let $q \in \mathbf{k}$ and $\pi \in S_n$. Then,

$$L_\pi^q = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) | i_j = i_{j+1}\}|} (q+1)^{|\{i_1, i_2, \dots, i_n\}|} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Furthermore

$$L_\pi^0 = L_\pi, \qquad L_\pi^1 = K_\pi$$

Properties of q-fundamental quasisymmetric functions

Definition (Enriched q-monomials)

Let $q \in \mathbf{k}$ and $I \subseteq [n-1]$ The enriched q-monomial indexed by I is defined as

$$\eta_I^q = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in I \Rightarrow i_j = i_{j+1}}} (q+1)^{|\{i_1, \ldots, i_n\}|} x_{i_1} \ldots x_{i_n}.$$

Proposition

Let $I \subseteq [n-1]$

$$\eta_I^q = \sum_{I \subseteq J} (q+1)^{n-|J|} M_J.$$

where M_J is the usual monomial quasisymmetric function

$$M_J = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in J \Leftrightarrow i_j = i_{j+1}}} x_{i_1} \ldots x_{i_n}.$$

Properties of q-quasisymmetric functions

Theorem (Grinberg, V. 2022)

Let $q \in \mathbf{k}$ be such that $q+1$ is invertible. The family of enriched q-monomial quasisymmetric functions $\left(\eta^q_{n,I}\right)_{n\geq 0, I\subseteq[n-1]}$ is a basis of QSym. *Furthermore*

$$(q+1)^{n-|J|} M_J = \sum_{J\subseteq I} (-1)^{|I\setminus J|} \eta^q_I .$$

Theorem (Grinberg, V. 2022)

Let $I \subseteq [n-1]$ and $q \in \mathbf{k}$. The q-fundamental quasisymmetric functions may be expressed in the enriched q-monomial basis as

$$L^q_I = \sum_{\substack{J\subseteq I \\ K\subseteq \mathrm{Peak}(I) \\ J\cap K=\emptyset}} (-q)^{|K|} (q-1)^{|J|} \eta^q_{J\cup(K-1)\cup K} .$$

Proposition (Antipode)

For an integer n *and a subset* $I \subseteq [n-1]$, *we set* $n - I = \{n - i | i \in I\}$. *The antipode of* QSym *can be defined as the unique* $\mathbf{k}$*-linear map* $S : \mathrm{QSym} \to \mathrm{QSym}$ *that satisfies*

$$S(M_I) = (-1)^{n-|I|} \sum_{n-I \subseteq J} M_J.$$

Let $q \in \mathbf{k}$ *invertible and* $p = 1/q$.

$$S(L_I^q) = (-q)^n L_{n-I}^p.$$

When the q-fundamental are a basis of QSym and new proper subalgebras

Are q-fundamental quasisymmetric functions a basis of QSym?

To know whether $(L^q_{n,I})_{n\geq 0, I\subseteq[n-1]}$ is a basis of QSym for some value of q appears as a natural question. For example, for $n=3$

- $\eta^q_{\emptyset} = L^q_{\emptyset}$;
- $(q-1)\eta^q_{\{1\}} = L^q_{\{1\}} - L^q_{\emptyset}$;
- $(q-1)\eta^q_{\{2\}} = \frac{(q-1)^2}{(q-1)^2+q}(L^q_{\{2\}} - L^q_{\emptyset}) + \frac{q}{(q-1)^2+q}(L^q_{\{1,2\}} - L^q_{\{1\}})$;
- $\eta^q_{\{1,2\}} = \frac{1}{(q-1)^2+q}\left(L^q_{\{1,2\}} - L^q_{\{2\}} - L^q_{\{1\}} + L^q_{\emptyset}\right)$.

We see that except for the case of Stembridge $q=1$, the cases $q\in\{e^{i\pi/3}, e^{-i\pi/3}\}$ (and the degenerate case $q=-1$), $\left(L^q_{3,I}\right)_{I\subseteq[2]}$ seems to be a basis.

Basis of QSym when q is not a root of unity.

Theorem (Grinberg, V. 2022)

Let $\mathbf{k} = \mathbb{C}$ be the set of complex numbers. The family of q-fundamental quasisymmetric functions $(L^q_{n,I})_{n\geq 0, I\subseteq[n-1]}$ is a basis of QSym for

$$q \notin \{e^{i2k\pi/l} | k, l \in \mathbb{N} \times \mathbb{P}\}.$$

To prove this theorem we characterise the transition matrix between the q-fundamental and enriched q-monomial quasisymmetric functions and show it is invertible when q is not a root of unity.

Basis of QSym when q is not a root of unity.

Definition

Let B_n be the transition matrix between $(L_I^q)_{I\subseteq[n-1]}$ and $(\eta_J^q)_{J\subseteq[n-1]}$.

Example

For $n = 4$, let us show the transition matrix B_4

	$\emptyset$	$\{1\}$	$\{2\}$	$\{2,1\}$	$\{3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,2,1\}$
$\emptyset$	1	0	0	0	0	0	0	0
$\{1\}$	1	q-1	0	0	0	0	0	0
$\{2\}$	1	0	q-1	-q	0	0	0	0
$\{2,1\}$	1	q-1	q-1	$(q\text{-}1)^2$	0	0	0	0
$\{3\}$	1	0	0	0	q-1	0	-q	0
$\{3,1\}$	1	q-1	0	0	q-1	$(q\text{-}1)^2$	-q	-$q(q\text{-}1)$
$\{3,2\}$	1	0	q-1	-q	q-1	0	$(q\text{-}1)^2$	-$q(q\text{-}1)$
$\{3,2,1\}$	1	q-1	q-1	$(q\text{-}1)^2$	q-1	$(q\text{-}1)^2$	$(q\text{-}1)^2$	$(q\text{-}1)^3$

Basis of QSym when q is not a root of unity.

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	$\emptyset$	$\{1\}$	$\{2\}$	$\{2,1\}$	$\{3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,2,1\}$
$\emptyset$	1	0	0	0	0	0	0	0
$\{1\}$	1	q-1	0	0	0	0	0	0
$\{2\}$	1	0	q-1	-q	0	0	0	0
$\{2,1\}$	1	q-1	q-1	$(q\text{-}1)^2$	0	0	0	0
$\{3\}$	1	0	0	0	q-1	0	-q	0
$\{3,1\}$	1	q-1	0	0	q-1	$(q\text{-}1)^2$	-q	-$q(q\text{-}1)$
$\{3,2\}$	1	0	q-1	-q	q-1	0	$(q\text{-}1)^2$	-$q(q\text{-}1)$
$\{3,2,1\}$	1	q-1	q-1	$(q\text{-}1)^2$	q-1	$(q\text{-}1)^2$	$(q\text{-}1)^2$	$(q\text{-}1)^3$

Basis of QSym when q is not a root of unity.

Definition

Let B_n be the transition matrix between $(L_I^q)_{I\subseteq[n-1]}$ and $(\eta_J^q)_{J\subseteq[n-1]}$.

Example

For $n = 4$, let us show the transition matrix B_4

	$\emptyset$	$\{1\}$	$\{2\}$	$\{2,1\}$	$\{3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,2,1\}$
$\emptyset$	1	0	0	0	0	0	0	0
$\{1\}$	1	q-1	0	0	0	0	0	0
$\{2\}$	1	0	q-1	-q	0	0	0	0
$\{2,1\}$	1	q-1	q-1	$(q$-$1)^2$	0	0	0	0
$\{3\}$	1	0	0	0	q-1	0	-q	0
$\{3,1\}$	1	q-1	0	0	q-1	$(q$-$1)^2$	-q	-$q(q$-$1)$
$\{3,2\}$	1	0	q-1	-q	q-1	0	$(q$-$1)^2$	-$q(q$-$1)$
$\{3,2,1\}$	1	q-1	q-1	$(q$-$1)^2$	q-1	$(q$-$1)^2$	$(q$-$1)^2$	$(q$-$1)^3$

Basis of QSym when q is not a root of unity.

- For each $k \in [n]$, let A_k denote the transition matrix from $(L_I^{(q)})_{I \subseteq [n-1],\ \max(I)=k-1}$ to $(\eta_J^{(q)})_{J \subseteq [n-1],\ \max(J)=k-1}$ (where $\max \varnothing := 0$); this actually does not depend on n.
- Note that A_k is a $2^{k-1} \times 2^{k-1}$-matrix if $k \geq 2$, whereas A_1 is a 1×1 matrix.

$$B_n = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ * & A_2 & 0 & \dots & 0 \\ * & * & A_3 & \dots & 0 \\ & & & \ddots & \\ * & * & * & \ddots & 0 \\ * & * & * & * & A_n \end{pmatrix}.$$

- The matrices $(B_n)_n$ and $(A_n)_n$ satisfy the following recurrence relations (for $n \geq 1$ and $n \geq 2$, repsectively):

$$B_n = \begin{pmatrix} B_{n-1} & 0 \\ B_{n-1} & A_n \end{pmatrix}, \quad A_n = \begin{pmatrix} (q-1)B_{n-2} & -qB_{n-2} \\ (q-1)B_{n-2} & (q-1)A_{n-1} \end{pmatrix}.$$

Basis of QSym when q is not a root of unity.

We obtain that B_n is invertible thanks to the computation of its determinant. Namely,

$$|B_n| = \prod_{i \leq n} |A_i|$$

with

$$|A_n| = (-1)^{n(n-1)/2} [n]_{-q}! \prod_{i=1}^{n-2} |B_i|.$$

and $[n]_{-q}!$ is a *q-factorial*. Namely:

$$[n]_{-q} = 1 - q + q^2 - \cdots + (-q)^{n-1},$$
$$[n]_{-q}! = [1]_{-q} \cdot [2]_{-q} \cdots [n]_{-q}.$$

Then, notice that $[n]_{-q}!$ is non 0 for all n iff $q \notin \{e^{i2k\pi/l} | k, l \in \mathbb{N} \times \mathbb{P}\}$.

What happens when q is a root of unity ?

- When q is a root of unity we find that a subfamily of the $L^{(q)}$ is the basis of a proper subalgebra of QSym.
- Start with the introduction of an *extended peak set*.

Definition (Extended peak set)

Let n and p be two positive integers. We say that $I \subseteq [n-1]$ is a p-extended peak set if $I \cup \{0\}$ does not contain more than p consecutive elements (as a result, $[1,p] \not\subseteq I$). We write $I \subseteq_p [n-1]$ for this statement.

Example

Set $n = 9$. One has $\{4,8\} \subseteq_1 [8]$, $\{1,4,5,8\} \subseteq_2 [8]$, $\{1,2,4,5,6,8\} \subseteq_3 [8]$. However $\{1,2,4,5,8\} \not\subseteq_2 [8]$ as the subsequence $[1,2]$ containing 1 has size $2 \geq p$.

Extended peak sets as a permutation statistic

- Extended peak sets look like an **intermediary statistic** between peak and descent sets.
- Any peak set is a 1-extended peak set, and any descent set on permutations of n elements is a p-extended peak set for $p \geq n$.
- Given a permutation π in S_n and an integer $p \geq 1$, we can define $\mathrm{Peak}_p(\pi)$ as

$$\begin{aligned}\mathrm{Peak}_p(\pi) = &\{i \in \mathrm{Des}(\pi) | i \leq p-1 \\ &\text{or } \exists\, 1 \leq j \leq p \text{ such that } i-j \notin \mathrm{Des}(\pi)\}.\end{aligned}$$

- On the one hand, $\mathrm{Peak}(\pi) = \mathrm{Peak}_1(\pi) \subseteq \mathrm{Peak}_p(\pi) \subseteq \mathrm{Des}(\pi)$.
- On the other hand, $\mathrm{Peak}_p(\pi) \subseteq_p [n-1]$.
- For instance let $\pi = 54163287$. We have $\mathrm{Peak}_1(\pi) = \mathrm{Peak}(\pi) = \{4,7\} \subseteq \mathrm{Peak}_2(\pi) = \{1,4,5,7\} \subseteq \mathrm{Peak}_3(\pi) = \{1,2,4,5,7\} = \mathrm{Des}(\pi)$.

Number of extended peak sets

We count the number of extended peak sets.

Proposition

Let $n, p \in \mathbb{P}$. Define $s_n^{(p)}$ to be the number of p-extended peak sets on n elements. Extend the definition with $s_0^{(p)} = 0$ for all positive p. One has

$$s_n^{(p)} = \begin{cases} 2^{n-1} & \text{if } n \leq p, \\ \sum_{k=0}^{p} s_{n-k-1}^{(p)} = \sum_{k=1}^{p+1} s_{n-k}^{(p)} & \text{if } n > p. \end{cases}$$

Remark

The sequence $(s_n^{(p)})_n$ is actually a Fibonacci sequence of order p.

Extended peak quasisymmetric functions

We proceed with the definition of subfamilies of q-fundamentals when **q is a root of unity**.

Definition (Extended-peak quasisymmetric functions)

*Let $n, p \in \mathbb{P}$, and let ρ_p be such that $-\rho_p$ is a **primitive (p+1)-th root of unity**. We take for example $\rho_p = e^{-i\pi(p-1)/(p+1)}$. Thus, $\rho_1 = 1$, $\rho_2 = e^{-i\pi/3}$, $\rho_3 = e^{-i\pi/2}, \ldots$. As a result, $(-\rho_p)^{p+1} = 1$ but $(-\rho_p)^j \neq 1$ for $1 \leq j < p+1$. Given a subset $I \subseteq [n-1]$, we define the **p-extended peak quasisymmetric function** indexed by I*

$$L^p_{n,I} = L^{(\rho_p)}_{n,I}.$$

The algebra of extended peaks

Definition (The algebra of extended peaks)

Define $\mathcal{P}^p \subseteq \mathrm{QSym}$ to be the subalgebra of QSym spanned by $(L^p_{n,I})_{n\geq 0, I\subseteq[n-1]}$, and let $\mathcal{P}^p_n \subseteq \mathrm{QSym}_n$ be its subspace composed of quasisymmetric functions of degree n (that is, the vector space spanned by $(L^p_{n,I})_{I\subseteq[n-1]}$). We call $\mathcal{P}^p$ the ***algebra of p-extended peaks****.*

- We remark that $\mathcal{P}^p$ **is actually a Hopf subalgebra of** QSym.
- The definition gives extended peak functions over all subsets. However, we know that they do not span QSym. As a result, for all $p \in \mathbb{P}$, the family $(L^p_{n,I})_{n\geq 0, I\subseteq[n-1]}$ **is not linearly independent** and some indices are redundant.
- Our goal is to characterise these set indices.

Linear dependence

For $n, p \in \mathbb{P}$, if a set I is not a p-extended peak set, then $L^p_{n,I}$ may be expressed in terms of other p-extended peak quasisymmetric functions.

Theorem (Extended peak functions over sets that are not p-extended peaks, Grinberg, V. 2023)

Let $n, p \in \mathbb{P}$ with $n \geq p+1$. Let i be an integer such that $0 \leq i \leq n-1-p$, and $J \subseteq [n-1]$ be a subset that satisfies $[i+1, i+p+1] \cap J = \emptyset$ and $i \in J \cup \{0\}$. Then,

$$[i+1, i+p] \cup J \not\subseteq_p [n-1]$$

as it contains either a sequence of $p+1$ consecutive elements or the sequence $[1, p]$. Notice further that any set that is not a p-extended peak set may be written as such. We have the equality

$$\sum_{I \subseteq [i+1, i+p]} (-1)^{|I|} L^p_{n, I \cup J} = 0.$$

The proof relies on combinatorial manipulations of the formula relating q-fundamental and enriched q-monomial quasisymmetric functions.

Subspaces dimension

Secondly, we can compute explicitly the dimension of $\mathcal{P}_n^p$ for $n, p \in \mathbb{P}$.

Theorem (Subspaces dimension, Grinberg, V. 2023)

Let $n, p \in \mathbb{P}$ be two positive integers. The dimension of $\mathcal{P}_n^p$ is equal to $s_n^{(p)}$, the number of p-extended peak sets on n elements:

$$\dim \mathcal{P}_n^p = s_n^{(p)}.$$

The proof uses the recurrence relations in the transition matrix.

Characterisation of the extended peak subalgebra of QSym

Combining the two previous theorems we characterise the subalgebra $\mathcal{P}^p$.

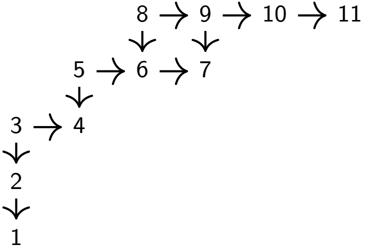
Theorem (Basis for the algebra of extended peaks, Grinberg, V. 2023)

Let $p \in \mathbb{P}$. The family $(L^p_{n,I})_{n \geq 0, I \subseteq_p [n-1]}$ is a basis of the subalgebra $\mathcal{P}^p$ of QSym.

The q-fundamental are the quasisymmetric expansion of Hall-Littlewood functions

Generating functions for Young diagrams

- Let $n \in \mathbb{P}$ and λ and μ be two partitions such that λ/μ is a skew shape and $|\lambda| - |\mu| = n$.
- Label the skew Young diagram of shape λ/μ with the successive integers of $[n]$ from left to right and bottom to top. Define the partial order $<_{\lambda/\mu}$ on $[n]$ as $i <_{\lambda/\mu} j$ if and only if i lies northwest of j and denote the labelled poset $P_{\lambda/\mu} = ([n], <_{\lambda/\mu})$.



- Consequently, the enriched $P_{\lambda/\mu}$-partitions are precisely the marked semistandard Young tableaux of shape λ/μ, i.e. $\mathcal{L}_{\mathbb{P}^{\pm}}(P_{\lambda/\mu}) = SSYT^{\pm}(\lambda/\mu)$

Theorem (Grinberg, V. 2024)

Let $n \in \mathbb{P}$ and λ/μ such that $|\lambda| - |\mu| = n$. The q-deformed generating function of $P_{\lambda/\mu}$ is exactly the Hall-Littlewood S-symmetric function with parameter $t = -q$.

$$S_{\lambda/\mu}(X; -q) = \Gamma^{(q)}([n], <_{\lambda/\mu}).$$

Relating Hall-Littlewood and q-fundammental

Theorem (Grinberg, V. 2024)

Let λ/μ be a skew shape. The Hall-Littlewood S-symmetric function with parameter $t = -q$ is related to q-fundamental quasisymmetric functions through

$$S_{\lambda/\mu}(X; -q) = \sum_{T \in SYT(\lambda/\mu)} L^{(q)}_{Des(T)}(X).$$

Theorem (Grinberg, V. 2024)

There is a ring homomorphism Θ_q: QSym $\Leftrightarrow \mathcal{P}^{(q)}$ such that for any positive integer n and any subset $I \subseteq [n-1]$, $\Theta_q\left(L^{(0)}_{n,I}\right) = L^{(q)}_{n,I}$. The restriction of Θ_q to Λ is θ_{-q} and the following ring map diagram is commutative:

$$\begin{array}{ccc} \mathrm{QSym} & \xrightarrow{\Theta_q} & \mathcal{P}^{(q)} \\ \uparrow & & \uparrow \\ \Lambda & \xrightarrow{\theta_{-q}} & \Lambda_{-q} \end{array}$$

Cauchy like formula

- $Y = \{y_1, y_2, \ldots,\}$ is an additional alphabet of indeterminates, independent of and commuting with X
- The product alphabet XY is $XY = \{x_i y_j\}_{i,j}$.
- Let $\pi \in \mathfrak{S}_n$. Extend the definition of $\Gamma^{(q)}$ to XY by considering P_π-partitions $(f,g) : i \mapsto (f(i), g(i))$ with values in $\mathbb{P} \times \mathbb{P}^{\pm}$ equipped with the lexicographic order.
- We say that $(i,j) \in \mathbb{P} \times \mathbb{P}^{\pm}$, is negative if and only if j is negative.
- We define

$$\Gamma^{(q)}(P_\pi)(XY) = \sum_{(f,g)\in\mathcal{L}_{\mathbb{P}\times\mathbb{P}^{\pm}}([n],<_\pi)} \prod_{1\le i\le n} q^{[g(i)<0]} x_{f(i)} y_{|g(i)|}$$

Cauchy like formula

Proposition

The q-fundamental indexed by π on the product of indeterminate XY satisfies

$$L_\pi^{(q)}(XY) = \Gamma^{(q)}(P_\pi)(XY) = \sum_{\sigma \circ \tau = \pi} L_\sigma^{(0)}(X) L_\tau^{(q)}(Y).$$

The proof is similar to a proof from Petersen in *Enriched P-partitions and peak algebras.*

Theorem (Grinberg, V. 2024)

Recall the following nice Cauchy like formula for Hall-Littlewood symmetric functions by Macdonald.

$$q_n(XY;t) = \sum_{\lambda \vdash n} s_\lambda(X) S_\lambda(Y;t).$$

This formula is a direct consequence of our results

Cauchy like formula

To prove this result:

- Specialise our proposition to get:

$$q_n(XY;-q) = L^{(q)}_{id_n}(XY) = \sum_{\sigma\in\mathfrak{S}_n} L^{(0)}_{\sigma^{-1}}(X)L^{(q)}_{\sigma}(Y),$$

  where $id_n \in \mathfrak{S}_n$ is the identity permutation.

- Use the RS correspondence to reindex the sum over standard Young tableaux.

$$q_n(XY;-q) = \sum_{\lambda\vdash n}\sum_{T,U\in SYT(\lambda)} L^{(0)}_{Des(T)}(X)L^{(q)}_{Des(U)}(Y)$$

$$= \sum_{\lambda\vdash n}\left(\sum_{T\in SYT(\lambda)} L^{(0)}_{Des(T)}(X)\right)\left(\sum_{U\in SYT(\lambda)} L^{(q)}_{Des(U)}(Y)\right)$$

- Use our quasimmetric expansion formula for Hall-Littlewood symmetric functions.

Thanks !