Longueur atomique, un pont entre combinatoire et théorie des nombres

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Joint work with Olivier Brunat and Thomas Gerber

2 avril 2026

Outline of the talk

I. Background on integer partitions

Definition

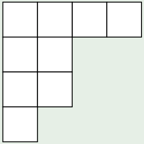
1. *A partition of n is a decreasing sequence of positive integers* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ *such that* $\sum \lambda_i = n$*. We write this sequence by* $[\lambda_1, \lambda_2, \ldots, \lambda_p]$*.*
2. *We have the Ferrer diagrams to represent the partitions.*

Answer

1. *A partition of n is a decreasing sequence of positive integers* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ *such that* $\sum \lambda_i = n$. *We write this sequence by* $[\lambda_1, \lambda_2, \ldots, \lambda_p]$.
2. *We have the Ferrer diagrams to represent the partitions.*

Example

Here is the Ferrer diagram of the partition $[4, 2, 2, 1]$ of 9.



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So, we have the beginning of a generating function :

$$q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

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Theorem (Euler)

$$\sum_{n=1}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \prod_{k=1}^{\infty}\frac{1}{1-q^k}.$$

Historical remark

Remark

Ramanujan was extremely interested in the numbers $p(n)$. He conjectured and proved that

$$p(5n+4) = 0 \textit{ mod } 5$$
$$p(7n+5) = 0 \textit{ mod } 7$$
$$p(11n+6) = 0 \textit{ mod } 11.$$

Definition of t-cores

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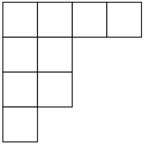
Let λ be a partition n. We say that λ is a t-core of size n if λ does not have any rim-hook of length t.



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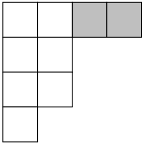

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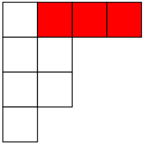

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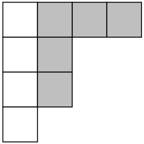

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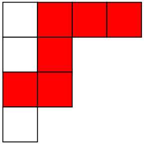

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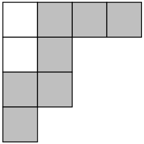

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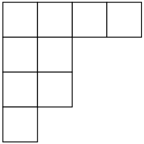

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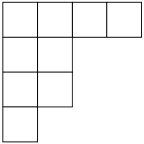
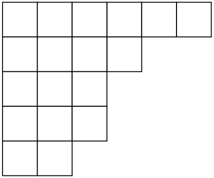


Conclusion : The partition $[4, 2, 2, 1]$ is a 6-core of size 9.

The peeling algorithm (Example for $t = 4$)

From any partition λ, we can peel it off by removing all the rim-hooks of length t. What we are left with is a t-core.



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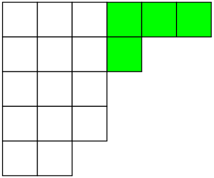

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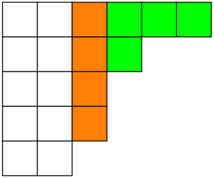

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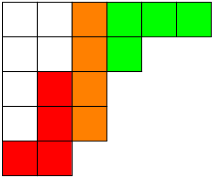

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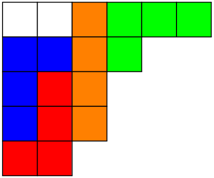

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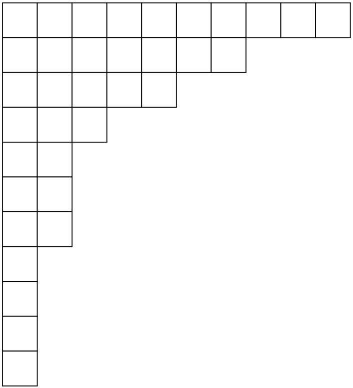

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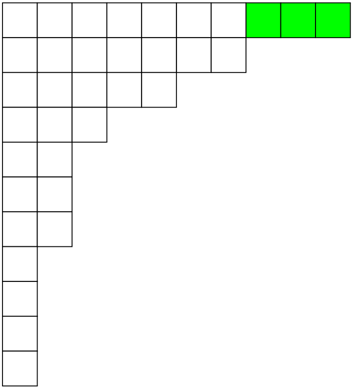

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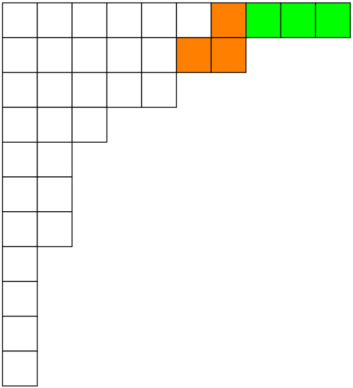

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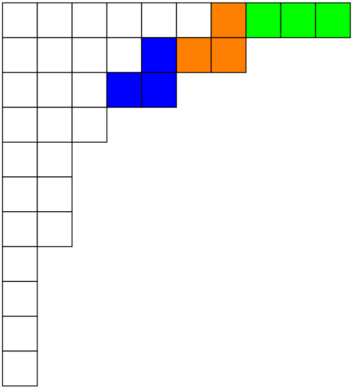

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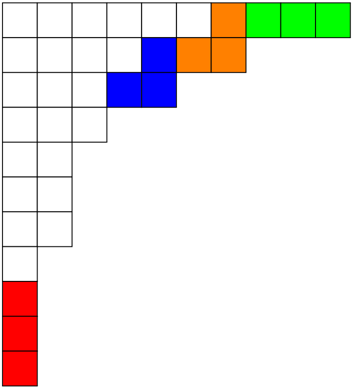

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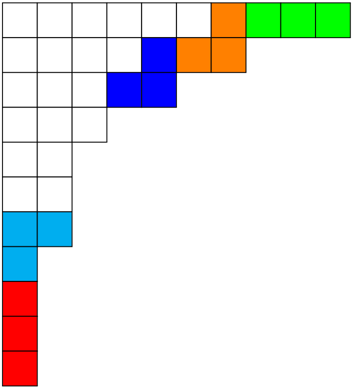

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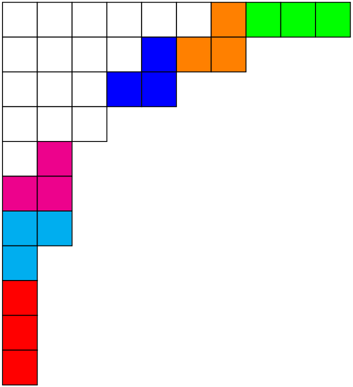

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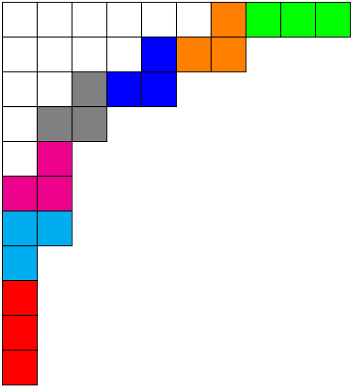

We cannot continue. The white part is then a 3-core of size 14.

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6. $n = 5$: $\{[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]\}$ $1q^5$

Another natural question

Question

Let $n \in \mathbb{N}$ and $t \geq 1$. What is the number of t-cores of size n ?

Example ($t = 3$)

1. $n = 0 : \emptyset$ $0q^0$
2. $n = 1 : \{[1]\}$ $1q^1$
3. $n = 2 : \{[2],[1,1]\}$ $2q^2$
4. $n = 3 : \{[3],[2,1],[1,1,1]\}$ $0q^3$
5. $n = 4 : \{[4],[3,1],[2,2],[2,1,1],[1,1,1,1]\}$ $2q^4$
6. $n = 5 : \{[5],[4,1],[3,2],[3,1,1],[2,2,1],[2,1,1,1],[1,1,1,1,1]\}$ $1q^5$

So, we also have the beginning of a generating function

$$q + 2q^2 + 2q^4 + q^5 + \dots$$

Question

Let $c_t(n)$ be the number of t-cores of size n.

(1) *What is the link between the generating functions*

$$\sum_{n=1}^{\infty} p(n)q^n = q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

$$\sum_{n=1}^{\infty} c_3(n)q^n = q + 2q^2 + 0q^3 + 2q^4 + 1q^5 + \dots$$

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(2) *More generally, what is the link between the generating functions*

$$\sum_{n=1}^{\infty} p(n)q^n \quad \textit{and} \quad \sum_{n=1}^{\infty} c_t(n)q^n \quad ?$$

A mysterious factorization

$$q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots = (?) \times (q + 2q^2 + 0q^3 + 2q^4 + 1q^5 + \ldots)$$

A mysterious factorization

$$q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots = \frac{1}{\prod_{k=0}^{\infty}(1 - q^{3k+3})^3} \times (q + 2q^2 + 0q^3 + 2q^4 + 1q^5 + \ldots)$$

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\begin{aligned}
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&= \frac{1}{(q^3; q^3)_\infty^3} \times (q + 2q^2 + 0q^3 + 2q^4 + 1q^5 + \ldots)
\end{aligned}
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and more generally

Theorem (Garvan-Kim-Stanton, 90')

For any $t \geq 0$ we have

$$\sum_{n=1}^{\infty} p(n)q^n = \frac{1}{(q^t; q^t)_\infty^t} \sum_{n=1}^{\infty} c_t(n)q^n.$$

The (former) t-core conjecture

Theorem (Granville-Ono, 96')

If $t \geq 4$ then $c_t(n) > 0$.

Remark

It is actually a difficult question in general, for $t \geq 4$ and $n \in \mathbb{N}$, to find the t-cores of size n. By G-O we know that we always have at least one but we don't have a general way of building them.

To end on the motivations

Question

Why do we care about t-cores ?

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Why do we care about t-cores ?

Answer

(1) *It is well known that* $Irr(\mathbb{C}\mathfrak{S}_n) \simeq \{partitions\ of\ n\}$.

(2) *The story is much more complicated for modular representations, that is when the field* $\mathbb{C}$ *is replaced by a field of characteristic* $p > 0$. *In this situation, the notion of t-core plays a crucial role. The t-cores are in bijection with the blocks, and the notion of block is important.*

II. Background on Coxeter groups

Definition of Coxeter groups

Definition

A Coxeter group is a pair (W, S) where W is a group and $S \subset W$ is a set of generators, with the presentation

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = e \rangle$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. The cardinality of S is called the rank of (W, S).

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Example

If we take the symmetric group $W = \mathfrak{S}_n$ then the adjacent transpositions $S := \{(i, i+1) \mid i = 1 \ldots, n-1\}$ generate W. The pair $(\mathfrak{S}_n, S)$ is a Coxeter group of rank $n - 1$.

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- Coxeter groups (non necessarily finite) were then introduced in 1934 by H. S. M. Coxeter as abstractions of real reflections groups.

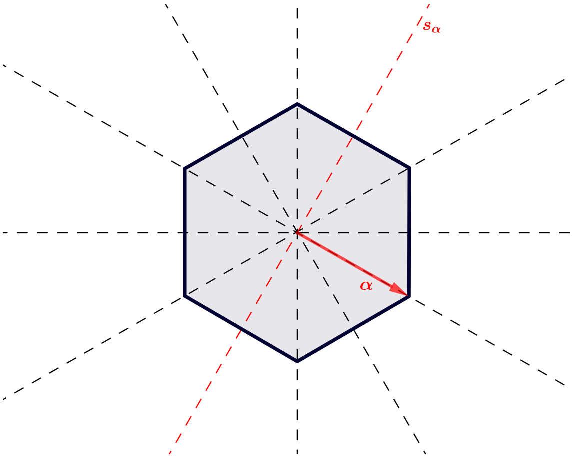

Figure – Here is a 6-gone. If we take the collection of the reflections associated to the hyperplanes on the figure then we get a subgroup of $O(\mathbb{R}^2)$ that is Coxeter group, namely the dihedral group D_6.

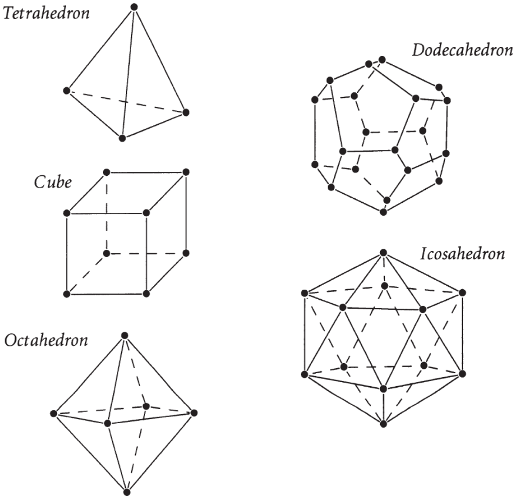

Figure – The symmetry group of each Platonic solid is a Coxeter group.

Words in Coxeter groups

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Coxeter length via inversion set

Definition

Let (W, S) be a Coxeter group with length function ℓ. Let Φ be a root system of W with $\Phi = \Phi^+ \sqcup \Phi^-$ its usual decomposition into positive roots and negative roots. To any root $\alpha \in \Phi^+$ one can associate a reflexion $s_\alpha \in W$ and this map a bijection.
Let $w \in W$. The inversion set of w is

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Proposition

Let $w \in W$. We have

$$\ell(w) = |\Phi(w)|.$$

Graphs of Coxeter groups

Goal : We want to represent Coxeter groups by means of graphs.

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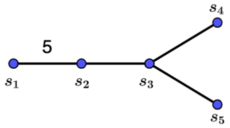
1) 

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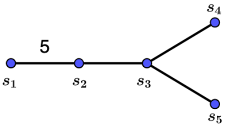

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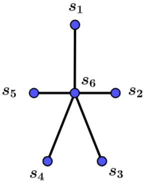

Quick classification of (irreducible) Coxeter groups

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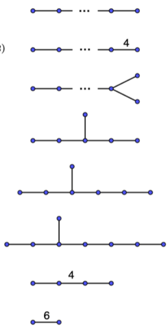

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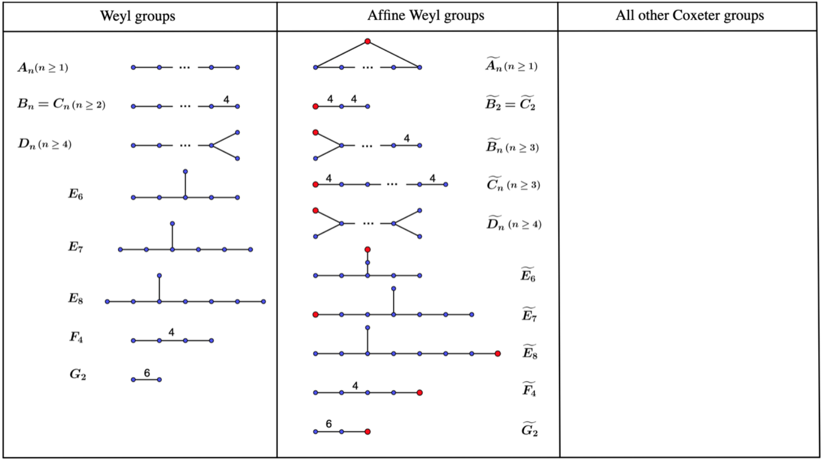

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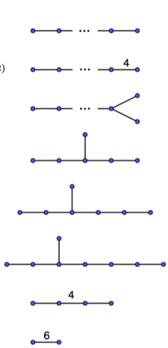
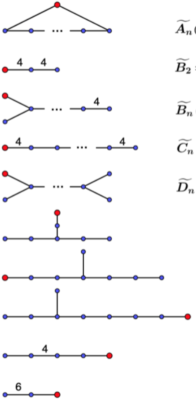
$A_n (n \geq 1)$ $B_n = C_n \ (n \geq 2)$ $D_n \ (n \geq 4)$ E_6 E_7 E_8 F_4 G_2 	 $\widetilde{A}_n \ (n \geq 1)$ $\widetilde{B}_2 = \widetilde{C}_2$ $\widetilde{B}_n \ (n \geq 3)$ $\widetilde{C}_n \ (n \geq 3)$ $\widetilde{D}_n \ (n \geq 4)$ $\widetilde{E}_6$ $\widetilde{E}_7$ $\widetilde{E}_8$ $\widetilde{F}_4$ $\widetilde{G}_2$	"This part is actually very big"

Geometrical construction of affine Weyl groups

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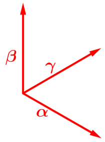
- Let $s_{\alpha,k}$ be the reflection associated to the hyperplane $H_{\alpha,k}$. We define the affine Weyl group W corresponding to Φ by

$$W := \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\rangle$$

where M is the coroot lattice.

Alcoves of $A_2^{(1)}$

Example : Let $\Phi^+ = \{\alpha, \beta, \gamma = \alpha + \beta\}$ be a positive root system of A_2.



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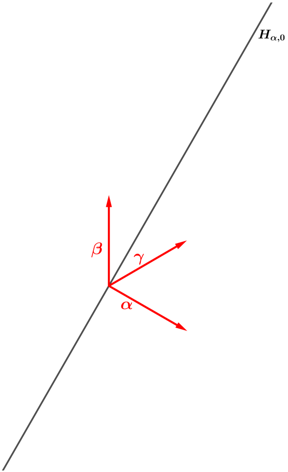

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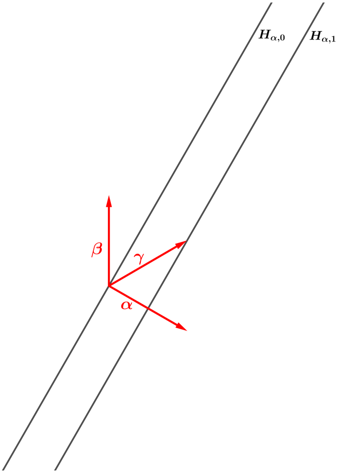

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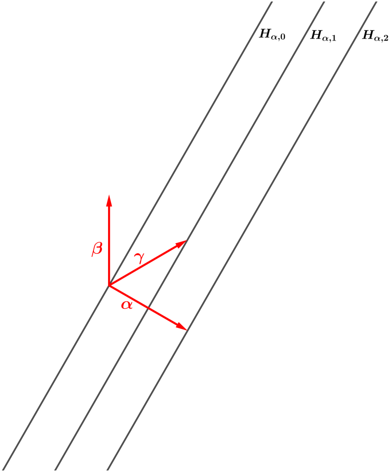

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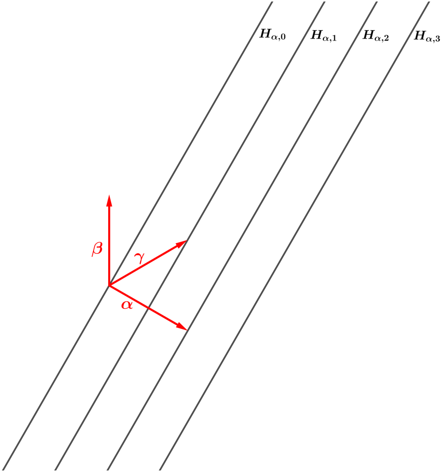

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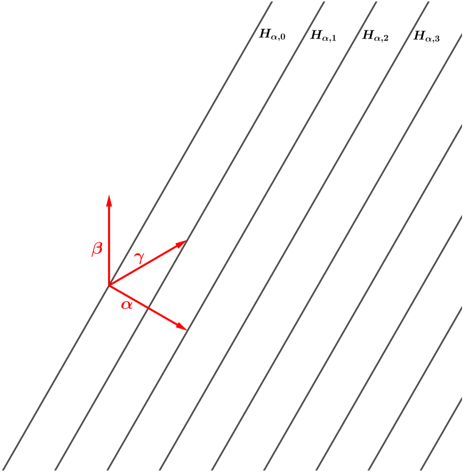

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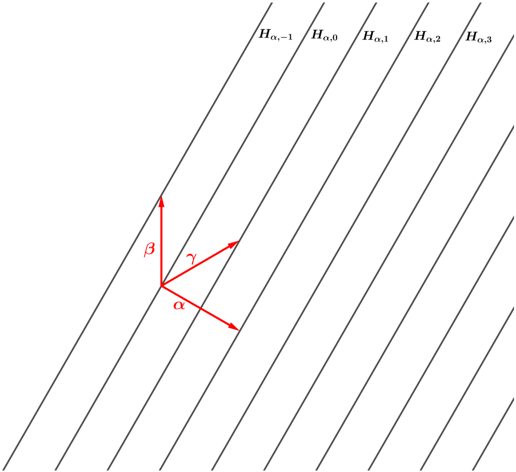

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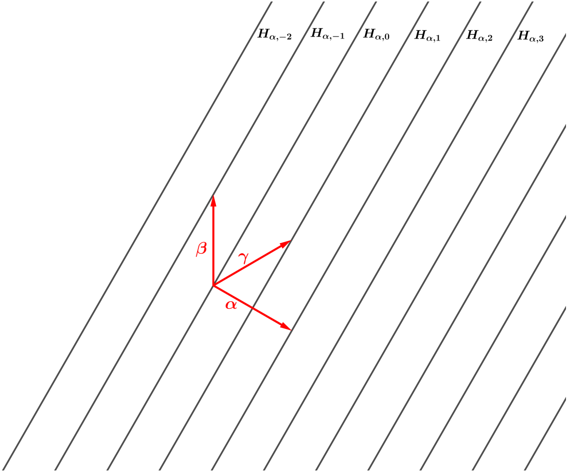

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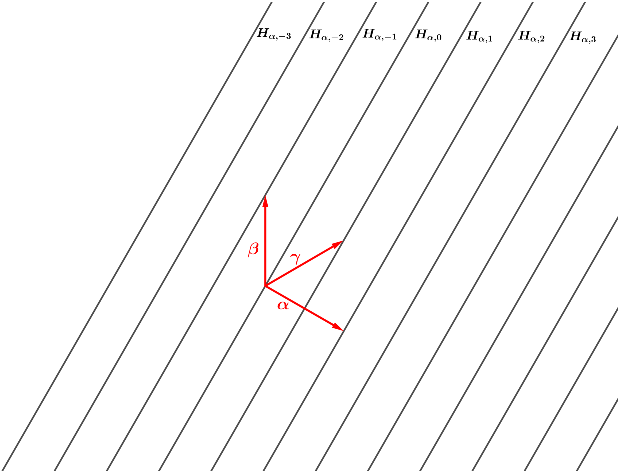

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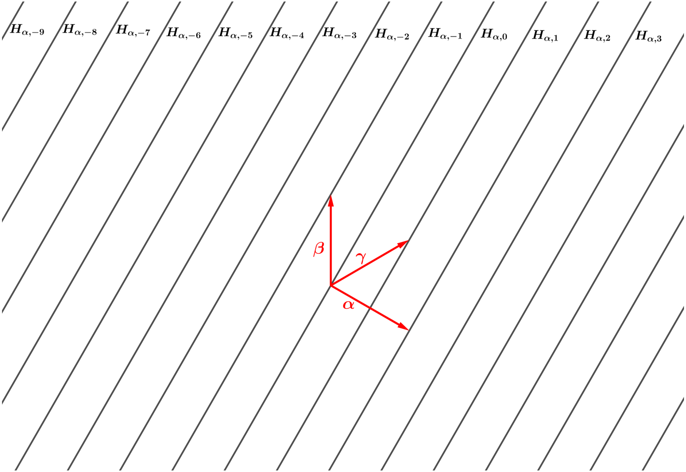

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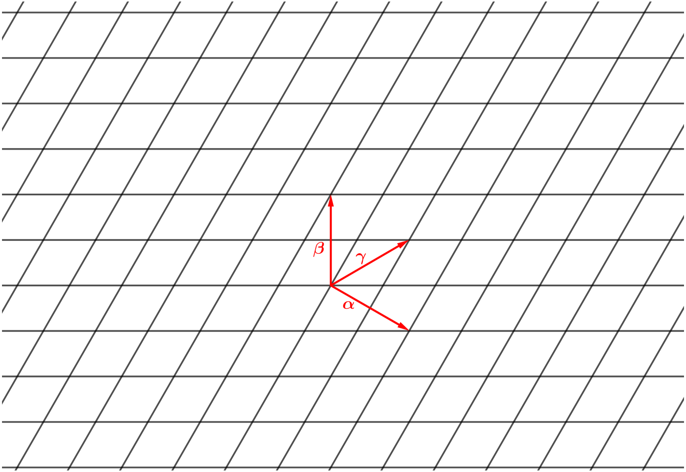

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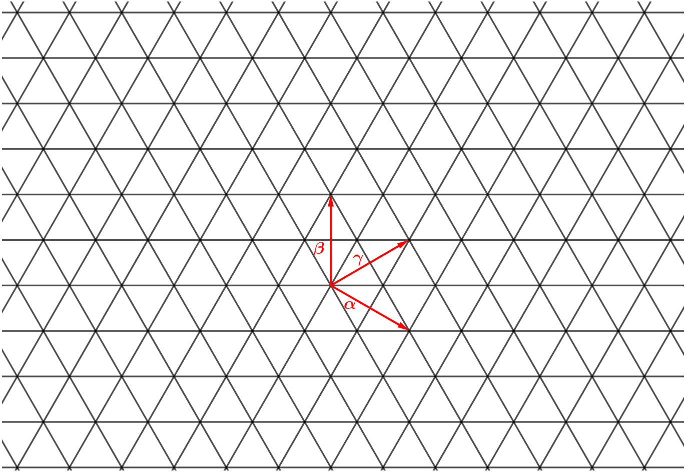

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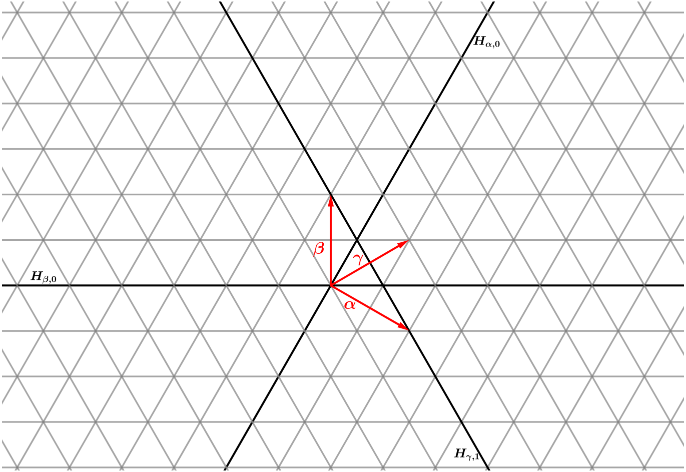

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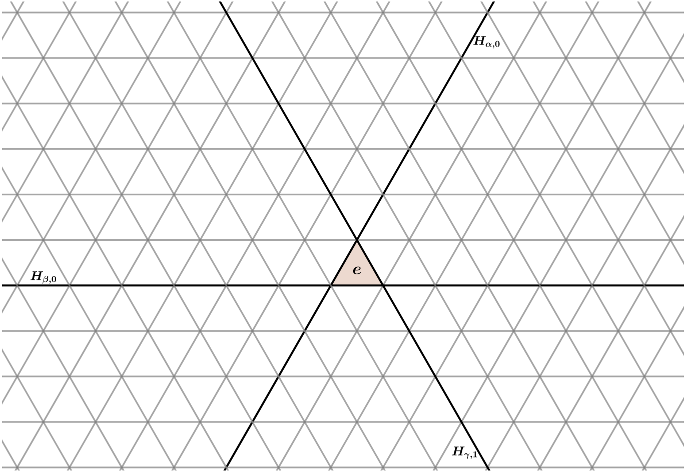

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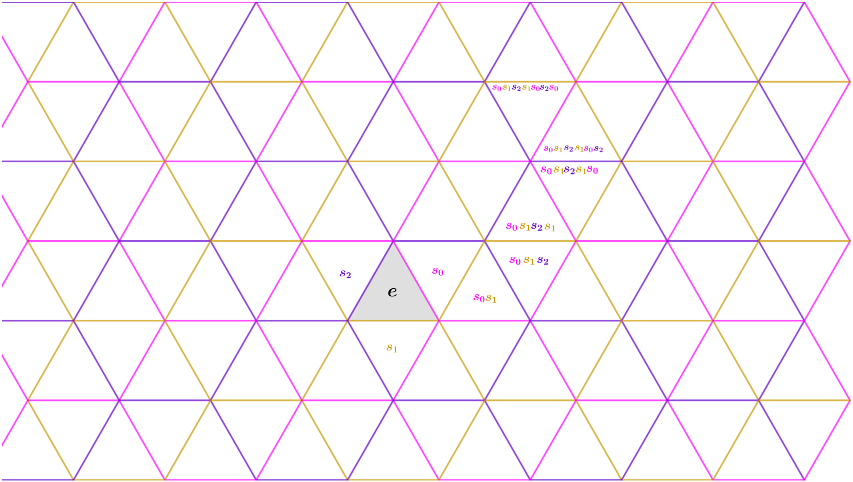
Question : Since W is a Coxeter group, is there a way to see the combinatorial relations in V ?

Relations in $A_2^{(1)}$ using alcoves



III. The atomic length

Main definition

Let $\mathfrak{g}$ be an affine Kac-Moody algebra, $\mathfrak{h}$ a Cartan subalgebra, $\langle -, - \rangle$ the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ and W the Weyl group of $\mathfrak{g}$. Let $\{\Lambda_0^\vee, \Lambda_1^\vee, \ldots, \Lambda_n^\vee\}$ be the set of affine fundamental coweigts and $\rho^\vee := \sum_{i=0}^{n} \Lambda_i^\vee$. Finally, let P be the affine weight lattice and P^+ be the set of affine dominant weights.

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Definition (CL-Gerber, 2022)

Let $\Lambda \in P^+$. The Λ-atomic length is

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Remark

We have two degrees of freedom to play with the definition

(1) *The weight Λ.*

(2) *The restriction of $\mathscr{L}_\Lambda$ to specific subgroups of $\mathrm{GL}(\mathfrak{h}^*)$.*

Let us write $q = \sum_{i=1}^{n} a_i \alpha_i$ in the simple basis $\{\alpha_1, \ldots, \alpha_n\}$ with $a_i \in \mathbb{R}$. The height of q is

$$ht(q) = a_1 + \cdots + a_n.$$

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By specialising $\Lambda = \rho := \sum_{i=1}^{n} \omega_i$ (which is an important weight of the theory) on the finite Weyl group W_0 we have

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This can be seen as a refinement of the usual length function since

$$\ell(w) = |\Phi(w)| = \sum_{\alpha \in \Phi(w)} 1.$$

The affine atomic length and Lascoux's bijection

Any element $w \in W$ can be written as a product of a translation $t_q \in T(M)$ and an element $w_0 \in W_0$, that is

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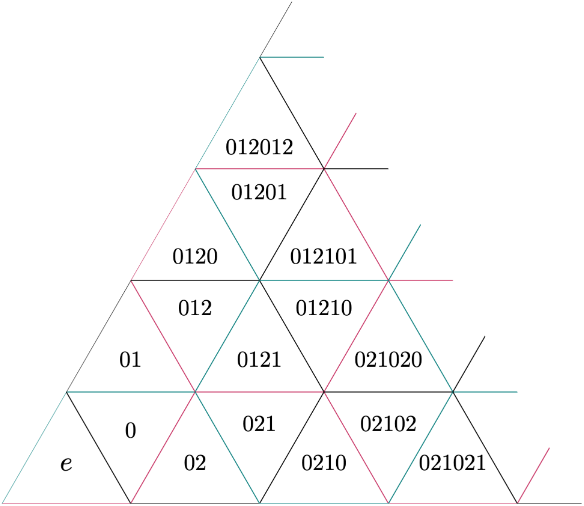
Theorem (CL-Gerber, 22')

Let W be the affine Weyl group of type $A_n^{(1)}$, let $q \in M$ and let $t_q \in W$ be the corresponding translation. One has

$$\mathscr{L}_{\Lambda_0}(t_q) = \text{size of the } (n+1)\text{-core associated to } q.$$

Example of Lascoux's bijection in type $A_2^{(1)}$



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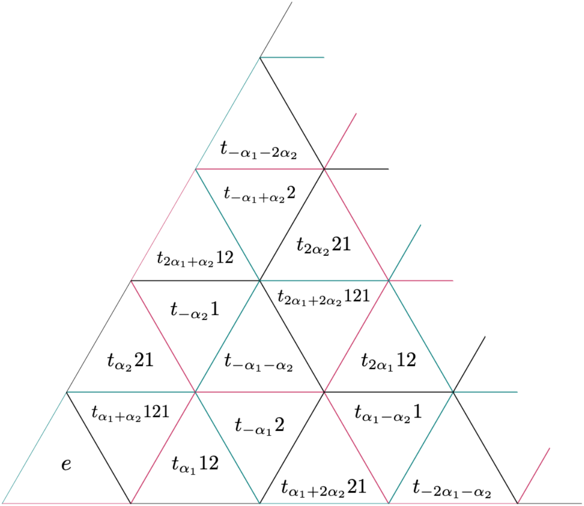

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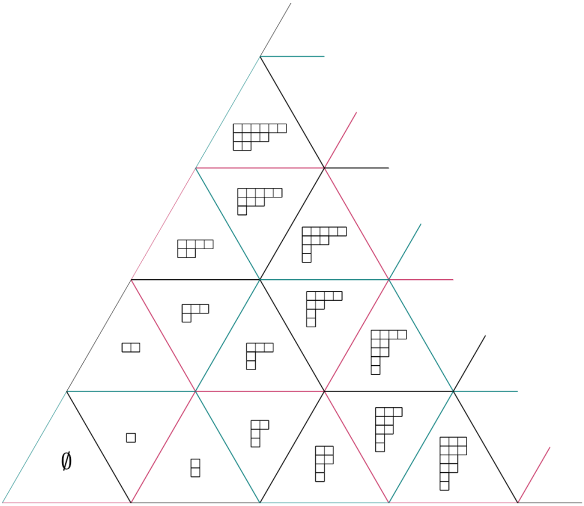

Link with diophantine equations

Theorem (CL-Gerber, 2022)

Let $w = t_q w_0 \in W$ with t_v the translation associated to v and w_0 the finite part of w. Let $ht(q)$ be the height of q. We have

$$\mathscr{L}_{\Lambda_\mathbf{0}}(w) = \frac{h}{2}||q||^2 - ht(q).$$

Example in type $A_3^{(1)}$

Let $w = t_q w_0 \in W$ with $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$. By the above theorem we have

$$\mathscr{L}_{\Lambda_0}(w) = 4(q_1^2 + q_2^2 + q_3^2 + q_1 q_2 + q_1 q_3 + q_2 q_3) - (3q_1 + 2q_2 + q_3).$$

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By doing a specific quadratic Gauss reduction on $\mathscr{L}_{\Lambda_0}(w)$ we get

$$\mathscr{L}_{\Lambda_0}(w) = \frac{1}{48}(12q_2 + 4q_3 - 1)^2 + \frac{1}{24}(8q_3 + 1)^2 + \frac{1}{16}(8q_1 + 4q_2 + 4q_3 - 3)^2 - \frac{5}{8}.$$

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We want then to consider the following equation

$$x^2 + 2y^2 + 3z^2 = 48N + 30.$$

The PIG theorem

Let G_{Λ_0} be the group defined by

$$G_{\Lambda_0} = \langle \frac{1}{2}\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}; s_{z=0} \rangle.$$

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Theorem (Brunat-CL-Gerber, 24')

Let Σ be the fundamental group of W. For any $\sigma \in \Sigma$ and any $w \in W$ we have

$$\mathscr{L}_{\Lambda_0}(\sigma w) = \mathscr{L}_{\Lambda_0}(w).$$

Theorem (Brunat-CL-Gerber, 24')

Let L be the coweight lattice. Let X be an integral solution of the Diophantine equation $x^2 + 2y^2 + 3z^2 = 48N + 30$. There exists $q \in L$ and $g \in G_{\Lambda_0}$ such that

$$g\varphi(q) = X.$$

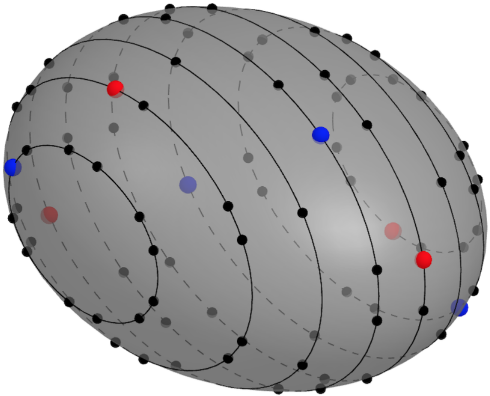

Figure – Integral solutions of $x^2 + 2y^2 + 3z^2 = 48 \cdot 2 + 30$, that is for $N = 2$.

Construction of 4-cores of any size

Corollary (Brunat-CL-Gerber, 24')

From any integral solution of the equation $x^2 + 2y^2 + 3z^2 = 48N + 30$, *one can construct a 4-core of size* N.

Merci