

Marche de l'éléphant, limite et urnes

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1. The elephant random walk

2. An elephant in a tree ?

- › Random recursive tree and Bernoulli percolation
- › News the superdiffusive limit (1)

3. An elephant inside an urn ?

- › Generalized Pólya urns
- › Fixed-point equation
- › News the superdiffusive limit (2)

The elephant random walk

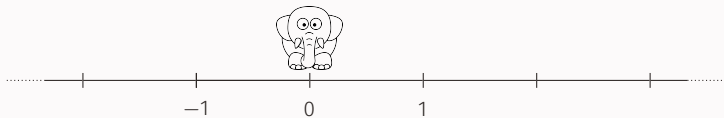
THE ELEPHANT RANDOM WALK

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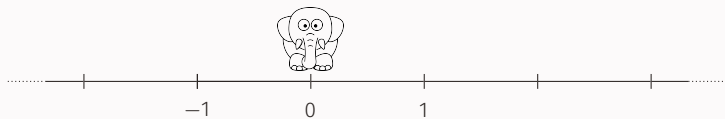
At time $n = 0$



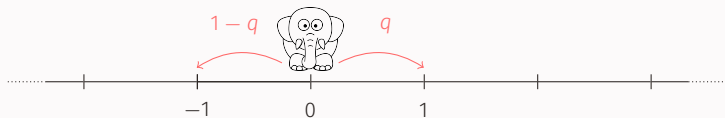
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At time $n = 1$



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$$X_{n+1} = \begin{cases} +X_k & \text{with probability } p, \\ -X_k & \text{with probability } 1 - p. \end{cases}$$

The position of the elephant is given by

$$S_{n+1} = S_n + X_{n+1}.$$

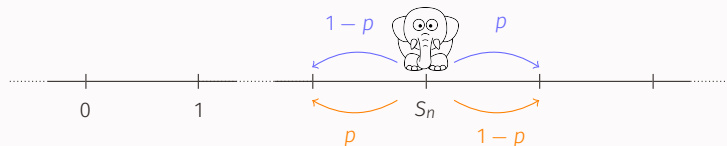
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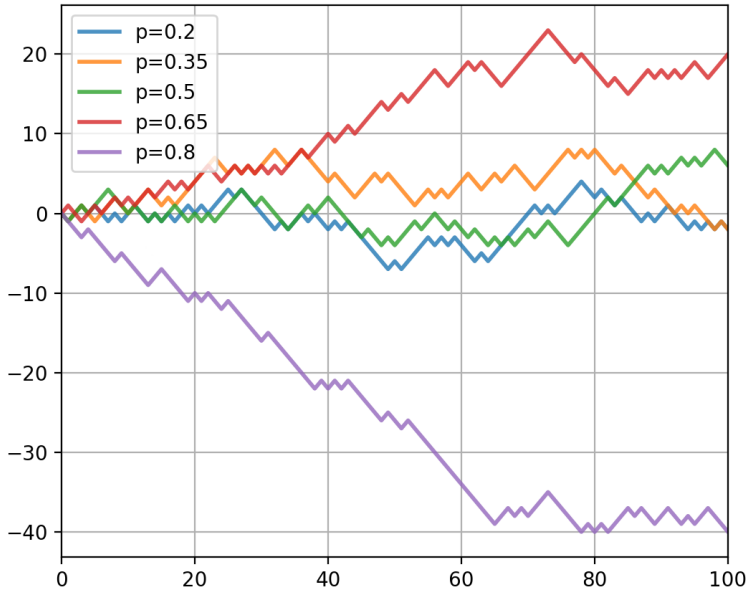
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$(X_k = -1)$

$(X_k = +1)$

A FEW TRAJECTORIES



It follows from the definition of the process

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}|\mathcal{F}_n] = S_n + (2p - 1)\frac{S_n}{n} = \left(1 + \frac{a}{n}\right)S_n = \gamma_n S_n,$$

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which ensures that (M_n) is a martingale

$$M_n = a_n S_n \quad \text{where} \quad a_1 = 1, \quad a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)}$$

with $\langle M \rangle_n = v_n + o(v_n)$ where $v_n = \sum_{k=1}^n a_k^2$.

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- › the superdiffusive regime where $a > 1/2$ (or $p > 3/4$) and $v_n = O(1)$.

Coletti et al. '17, Bercu '18, Kubota and Takei '19

Theorem (Law of large numbers)

Diffusive

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} \stackrel{\text{a.s.}}{=} 0$$

Critical

$$\lim_{n \rightarrow \infty} \frac{S_n}{\sqrt{n \log n}} \stackrel{\text{a.s.}}{=} 0$$

Superdiffusive

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^a} \stackrel{\text{a.s.}/\mathbb{L}^d}{=} L_q$$

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Theorem (Asymptotic normality)

Diffusive

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2a}\right)$$

Critical

$$\frac{S_n}{\sqrt{n} \log n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1)$$

Superdiffusive

$$\frac{S_n - n^a L_q}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2a-1}\right)$$

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Theorem (Distributional convergence (Baur and Bertoin, 2016))

Diffusive

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{} (W_t, t \geq 0)$$

where W_t is a centered gaussian process

$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} t^a s^{1-a}, \quad 0 < s \leq t$$

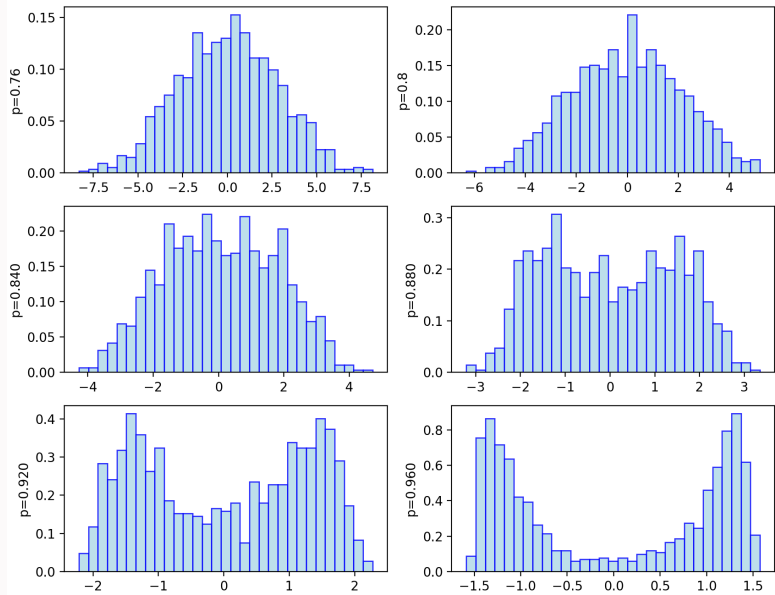
Critical

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n^t \log n}}, t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{} (B_t, t \geq 0)$$

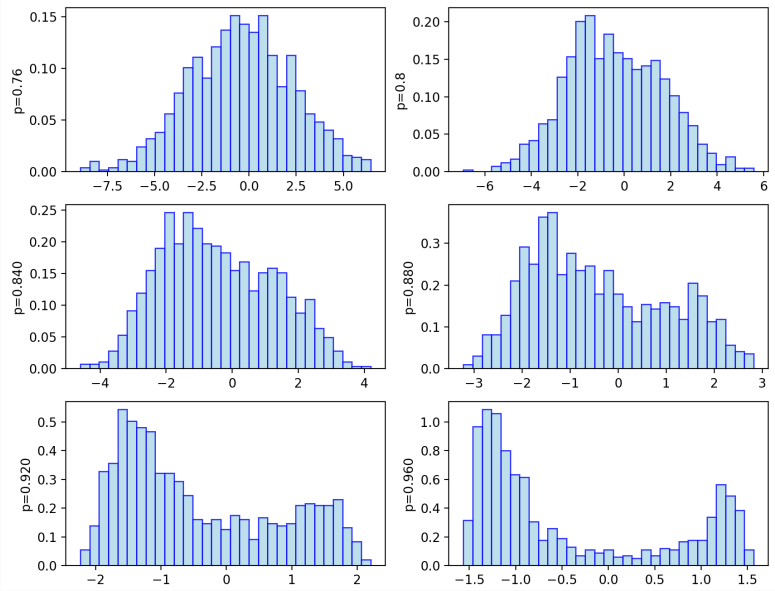
Superdiffusive

$$\left(\frac{S_{\lfloor nt \rfloor}}{n^a}, t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{} (t^a L_q, t \geq 0)$$

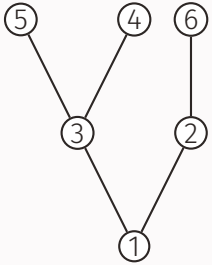
HISTOGRAM OF L_Q WHEN $Q=0.5$



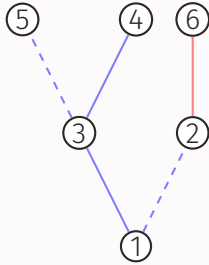
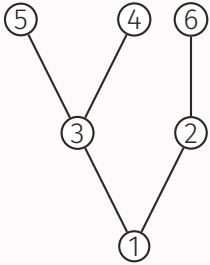
HISTOGRAM OF L_Q WHEN $Q=0.3$



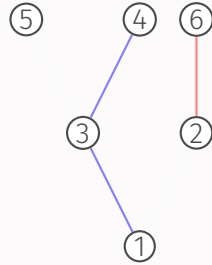
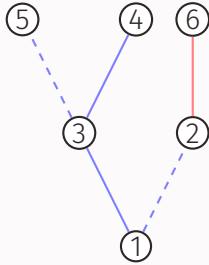
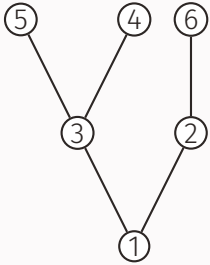
An elephant in a tree ?



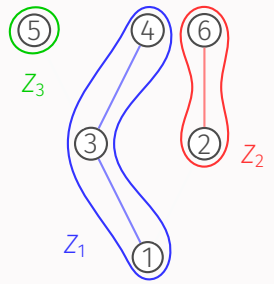
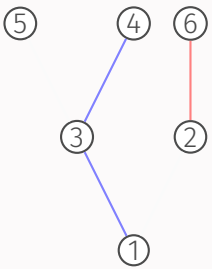
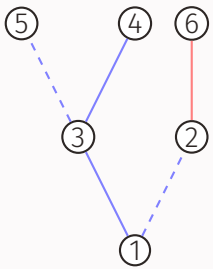
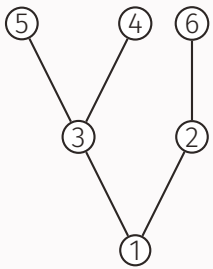
RRT AND BERNOULLI PERCOLATION



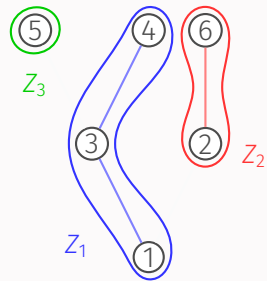
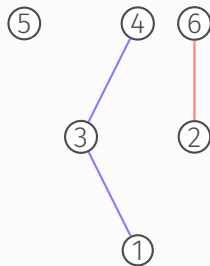
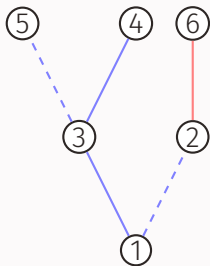
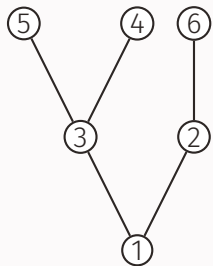
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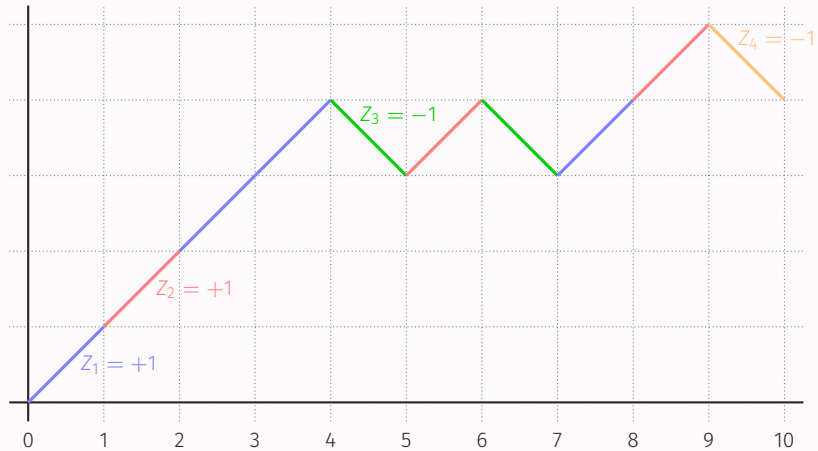
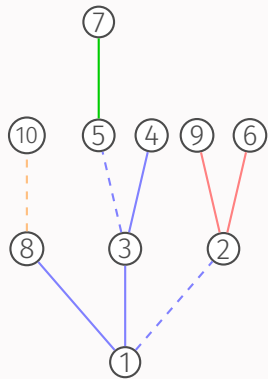
RRT AND BERNOULLI PERCOLATION



For $a > 0$ (i.e. $p > 1/2$),

$$S_n = \sum_{k=1}^n |C_n(k)| Z_k.$$

CAN ELEPHANTS CLIMB TREES ?



Thanks to Baur and Bertoin ('16) it is possible to show that the following decomposition of L_q holds :

The distribution of the superdiffusive limit

$$L_q = \sum_{i=1}^{\infty} C_i Z_i = \sum_{i=1}^{\infty} C_1^{(i)} (\beta_{\tau_i})^a Z_i$$

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where

- › $Z_1 \sim \mathcal{R}(q)$ and (Z_i) are i.i.d. random variables with $\mathcal{R}(1/2)$ distribution,
- › C_1 has a Mittag-Leffler distribution with parameter a ,
- › C_i a random variable with the same law as $(\beta_{\tau_i})^a C_1$,
- › $(\tau_i - 1)$ are dependent negative binomial random variables with respective parameters $(i - 1, 1 - a)$, such that $\tau_i = \tau_{i-1} + G_i$, where G_i has $\mathcal{G}(1 - a)$ distribution and is independent of τ_{i-1} ,
- › β_i denotes a beta variable with parameter $(1, i - 1)$ and is further independent of C_1 .

An elephant inside an urn ?

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What about $0 < p < 1$?

S. Janson – Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Processes and their Applications* 110 (2004)

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E. Baur and J. Bertoin – Elephant random walks and their connection to Pólya-type urns. *Physical review. E* 94 (2016).

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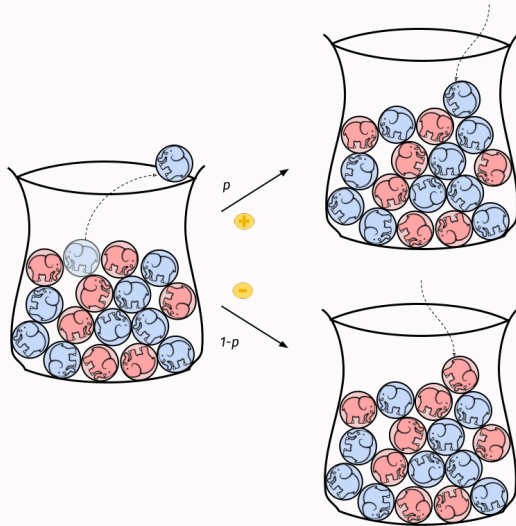
In this case the replacement matrix R becomes the **mean replacement matrix** A such that

$$A = \left(\mathbb{E}[\theta_{i,j}] \right)_{1 \leq i,j \leq q}$$

where q is the number of colors and $\theta_{i,j}$ is the random variable saying how many balls of type j are added when a ball of type i is picked.

ERW AND ASSOCIATED PÓLYA URN

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1	going right
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such that

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In this case, S_n has the same distribution as $R_n - B_n = 2R_n - n$.

Theorem (Janson, 2004)

When $a > 1/2$ and $U(0) = (\alpha, \beta)^T$, it holds

$$\lim_{n \rightarrow \infty} \frac{U(n) - nv_1}{n^a} = W_{(\alpha, \beta)} v_2 \quad \text{a.s.}$$

where $W_{(\alpha, \beta)}$ is a non-degenerate random variable such that $\mathbb{E}[W_{(\alpha, \beta)}] = \frac{\alpha - \beta}{\Gamma(1 + a)}$

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Moreover, we can show that $W_{(1,0)} \stackrel{\mathcal{L}}{=} -W_{(0,1)}$. Hence, it follows that

$$L_q \stackrel{\mathcal{L}}{=} Z_q W$$

where $Z_q \sim \mathcal{R}(q)$ is independent of $W := W_{(1,0)}$.

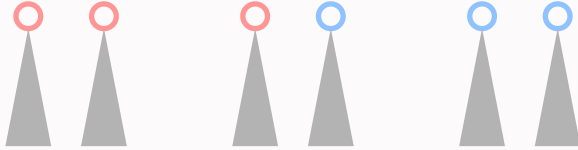
→ Joint work with **H. Guérin** (UQAM) and **K. Raschel** (U. Angers)

B. Chauvin, C. Mailler, N. Pouyanne – Smoothing equations for large Pólya urns. *J. Theoret. Probab* (2015)

with probability q ●

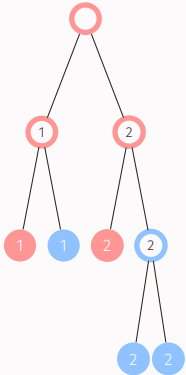
● with probability $1 - q$

URNS AND TREES



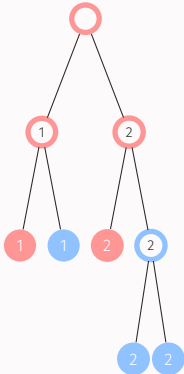
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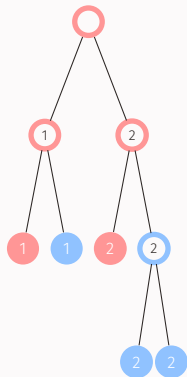
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At time n ,

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- › the number of drawings in the k -th subtree is $N_k(n) - 1$ (time inside).

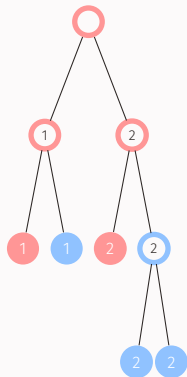




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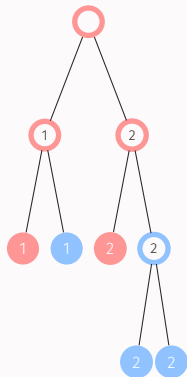


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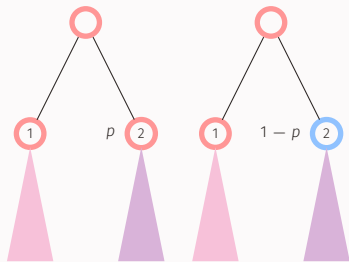
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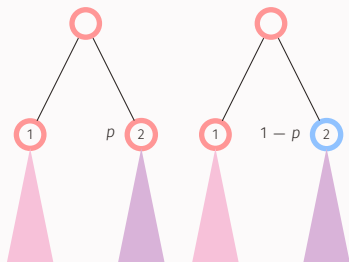
- › at each drawing in the k -th subtree, $N_k(n)$ increases by 1
- › $N(n) = (N_1(n), N_2(n))$ has exactly the same distribution as the 2-color Pólya urn process having I_2 as (deterministic) replacement matrix and $(1, 1)$ as initial composition.

Consider simultaneously



- › an urn process $N = (N_1, N_2)$ having I_2 as replacement matrix and $(1, 1)$ as initial condition,
 - › two urn processes $U_{(1,0)}^{(1)}$ and $U_{(1,0)}^{(2)}$ having A as mean replacement matrix and $(1, 0)$ as initial condition,
 - › an urn process $U_{(0,1)}^{(2)}$ having A as mean replacement matrix and $(0, 1)$ as initial condition,
 - › a Bernoulli random variable ξ_p with parameter p ,
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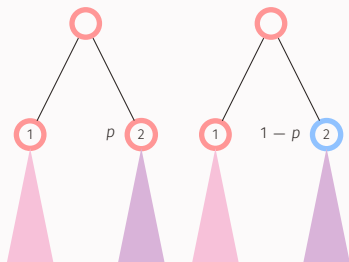


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Then, the process $U_{(1,0)} = (U_{(1,0)}(n))_n$ has the same distribution as

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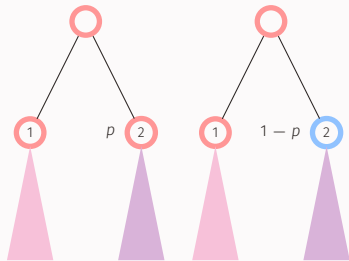
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DISTRIBUTIONAL EQUATION (1)

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which leads to

$$\begin{aligned} W_{(1,0)} &\stackrel{\mathcal{L}}{=} V^a W_{(1,0)}^{(1)} + \xi_p (1 - V)^a W_{(1,0)}^{(2)} + (1 - \xi_p) (1 - V)^a W_{(0,1)}^{(2)} \\ &\stackrel{\mathcal{L}}{=} V^a W_{(1,0)}^{(1)} + (2\xi_p - 1) (1 - V)^a W_{(1,0)}^{(2)}. \end{aligned}$$

Theorem

Let Wv_2 be the limit of a large two-color Pólya urn process with random replacement matrix A , initial composition $(0, 1)$ and ratio $a > 1/2$. Then,

$$W \stackrel{\mathcal{L}}{=} V^a W^{(1)} + Z_p (1 - V)^a W^{(2)}$$

where

- › V is a uniformly distributed random variable on $[0, 1]$,
- › Z_p is a Rademacher distributed random variable with parameter p ,
- › the $W^{(k)}$ are copies of W , all being independent of each other and of V and Z_p .

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Idea of proof. We want to show that $|\varphi_w(t)| \underset{\pm\infty}{=} O\left(\frac{1}{t^{k\rho}}\right)$ for any $k \in \mathbb{N}, \rho \in (0, 1/a)$.

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The random variable L_q satisfies Carleman's criterion and is moments-determined.

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Be careful: we cannot write $L_1 = C \cdot Y$ with Y independent of C and $\mathbb{E}[Y^k] = m_k$.

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By Kasahara's Tauberian theorem, we deduce:

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Let f be the density function of L_1 . We have:

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Moreover, there are explicit constants $c_+(a)$ and $c_-(a)$ such that:

$$f(x) \sim_{+\infty} c_+(a) x^{\frac{2a-1}{2(1-a)}} e^{-(1-a) \left(\frac{a^a}{\rho(a)x} \right)^{\frac{1}{1-a}}} \quad \text{and} \quad f(x) \sim_{-\infty} c_-(a) x^{\frac{2a^2-3a-1}{2(1-a^2)}} e^{-(1-a) \left(\frac{a^a}{\rho(a)x} \right)^{\frac{1}{1-a}}}$$

Thank you !

