The longest increasing subsequence of random separable permutations

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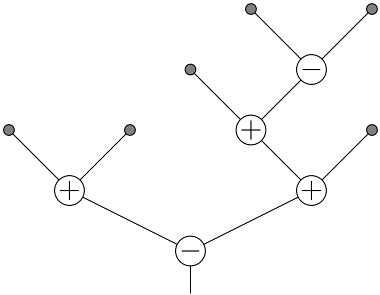
Joint work with Arka Adhikari, Jacopo Borga, William da Silva and Delphin Sénizergues

Signed binary trees



- Trees with n leaves (0 child) and $n-1$ nodes (2 children),
- distinction left child/right child,
- nodes are decorated with $+/-$ signs.

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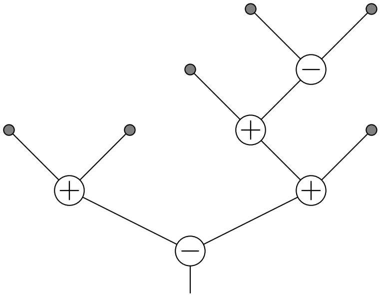

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- distinction left child/right child,
- nodes are decorated with $+/-$ signs.

- Very simple combinatorial structure : if τ_n counts binary trees with n leaves, grafting a leaf on an edge shows

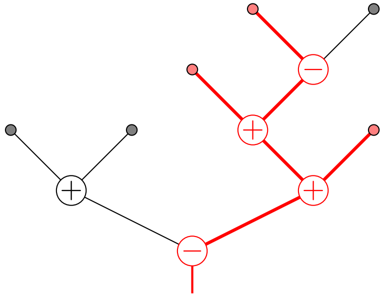
$$(n+1)\tau_{n+1} = 2(2n-1)\tau_n$$

so

$$\tau_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Positive subtrees

- A *positive subtree* is a subset A of the leaves such that for any $u, v \in A$, the highest common ancestor to u and v is $\oplus$.



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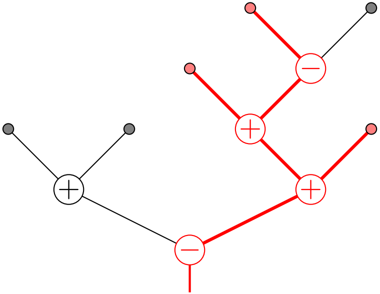

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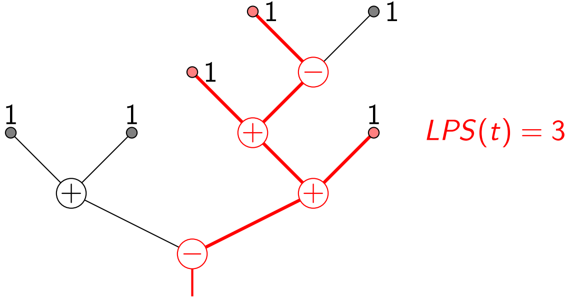

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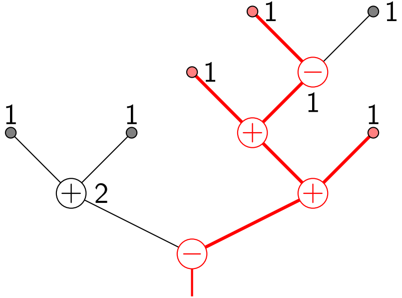

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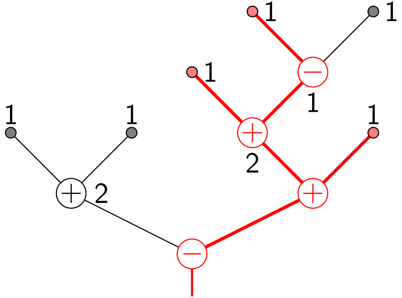

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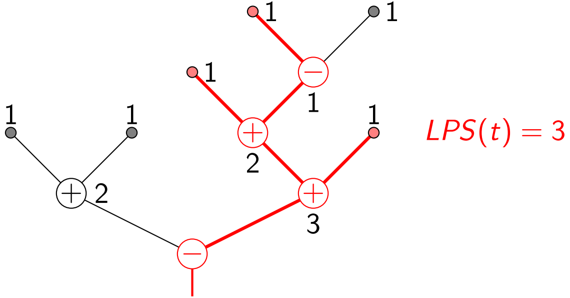

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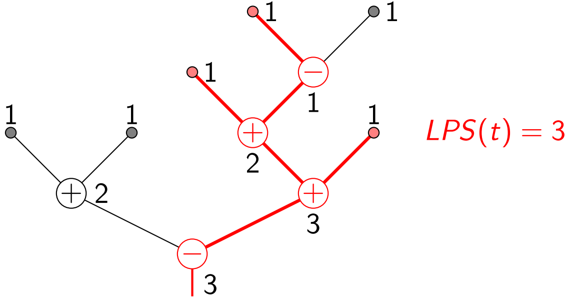

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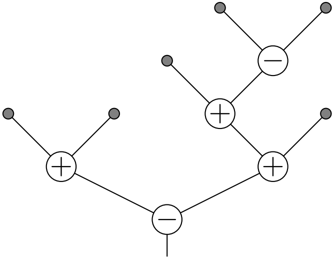
Probabilistic model

- Fix $p \in [0,1]$ and n a positive integer.
- Let T_n be chosen uniformly at random among binary trees with n leaves.
- Signs on the $n-1$ nodes are i.i.d. with $\mathbb{P}(\oplus) = p$ and $\mathbb{P}(\ominus) = 1-p$.
- What is the order of magnitude of $LPS(T_n)$ as $n \to +\infty$?

Motivation: random permutations

- Build a permutation from a signed binary tree:



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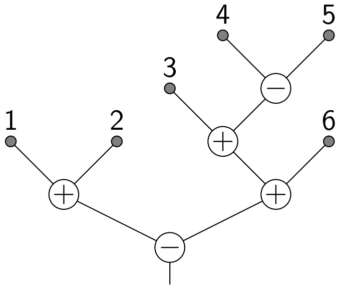

  - label leaves from left to right,

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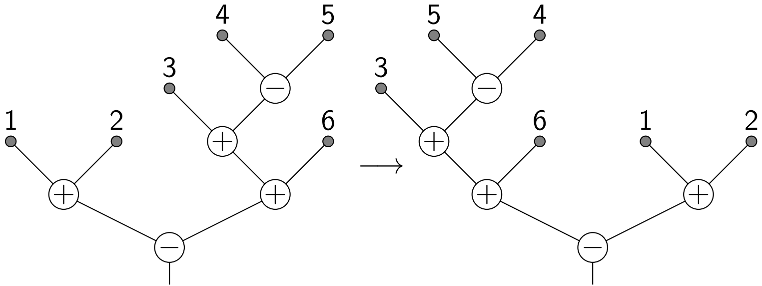

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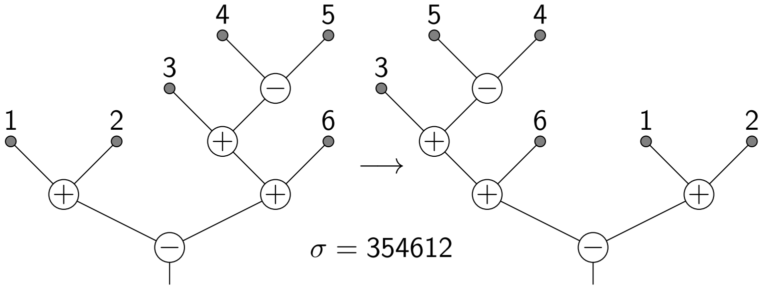

  - label leaves from left to right,
  - swap children at each $\ominus$ node,
  - read leaves from left to right.
- Permutations that can be obtained in this way are called *separable*.

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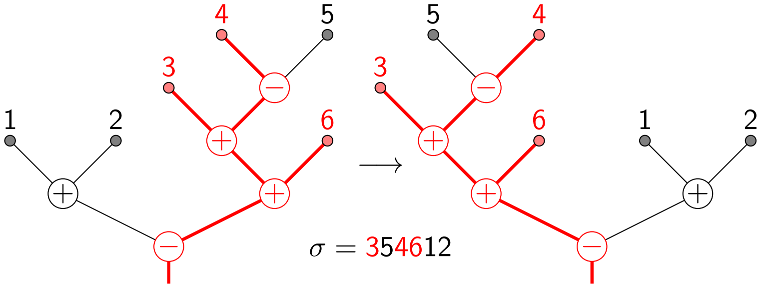
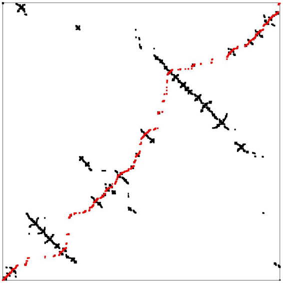


$\sigma = 354612$

- Equivalent definitions of separable permutations:
  - Permutations avoiding the patterns 2413 and 3142,
  - Permutations that can be obtained from n polynomials by ranking them according to their values at 0^+ and 0^-.
- $LPS(t)$ is also the longest increasing subsequence of σ.
- Also encodes the largest clique of cographs, and the longest oriented path of series-parallel graphs.
- The tree $\rightarrow$ permutation application is not bijective.

Motivation: random permutations

- The tree $\to$ permutation application is not bijective.
- Even for $p = \frac{1}{2}$, the permutation model is *not* uniform.
- Permutation sampled from the *Brownian separable permuton* μ_p with parameter p:
  - sample n i.i.d. points of $[0,1]^2$ from μ_p,
  - write $\sigma(i) = j$ if the i-th lowest point is also the j-th leftmost one.



- Then σ is the model we are studying!

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- $n^{\beta_*(p)+o(1)} \leq LPS(T_n) \leq n^{\beta^*(p)+o(1)}$ with $\frac{1}{2} < \beta_*(p) < \beta^*(p) < 1$ [Borga–da Silva–Gwynne 24].

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Theorem (Adhikari–Borga–B.–da Silva–Sénizergues 25)

We have the convergence in distribution

$$\frac{LPS(T_n)}{n^{\alpha(p)}} \xrightarrow[n\to+\infty]{(d)} X(p),$$

where $0 < X(p) < +\infty$ almost surely and $\alpha(p) = \frac{1}{2(\gamma(p)-1)} \in \left(\frac{1}{2}, 1\right)$, where $\gamma(p) \in \left(\frac{3}{2}, 2\right)$ is the solution to

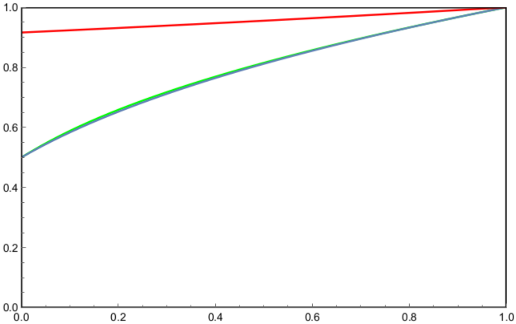
$$\frac{1 - p + 2^{\gamma-1}p}{\gamma - 1} = p\int_0^{1/2} x^{-\gamma}\left((1-x)^{-\gamma} - 1\right) \mathrm{d}x.$$

The exponent $\alpha(p)$

- The equation on α can be rewritten

$$\frac{1}{2^{1/\alpha}\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)} = \frac{p}{p-1}.$$

- Dependence in p: $\beta_*(p) < \alpha(p) < \beta^*(p)$.



- Fun fact: $\max_{p\in[0,1]} (\alpha(p) - \beta_*(p)) < \frac{1}{100}$.