

Effective Asymptotics of Combinatorial Systems

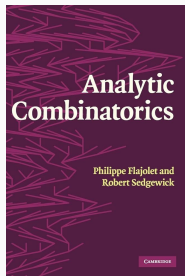
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joint work with Bruno Salvy

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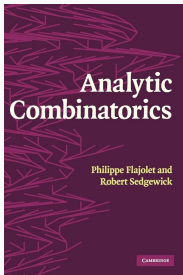
<https://arxiv.org/abs/2508.20008>



Analytic Combinatorics is a general approach:

- that starts from a **combinatorial description**
- translates this description into equations satisfied by **generating functions**
- views them as **analytic functions**
- and exploits their singular behavior to deduce **asymptotic** properties of the objects.

From book to algorithm

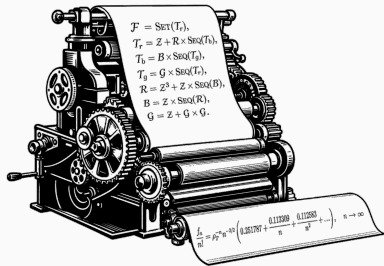


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What we did: develop computational tools that **automate** large parts of this approach.

Today: outline of the main steps.



First part: Combinatorics

Combinatorial systems

The symbolic method with a touch of species

Definition (Combinatorial system)

A *combinatorial system* $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ for an m -tuple $\mathcal{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ of *species* (labeled classes) is a vector

$$\begin{cases} \mathcal{Y}_1 = \mathcal{H}_1(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \\ \mathcal{Y}_2 = \mathcal{H}_2(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \\ \quad \vdots \\ \mathcal{Y}_m = \mathcal{H}_m(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \end{cases}$$

where each \mathcal{H}_i denotes a multisort species (**constructions**).

Definition (Specification)

A *specification* is a combinatorial system where the \mathcal{H}_i are combinations of

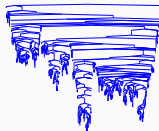
$$1, \mathcal{Z}, +, \times, \text{SEQ}, \text{CYC}, \text{SET}.$$

Let \mathcal{S} be a species,

- the size of a \mathcal{S} -structure is the number of \mathcal{Z} it contains;
- the number of \mathcal{S} -structures of any given size is finite.

Simple examples(?)

- Permutations: $\mathcal{P} = \text{SET}(\text{CYC}(\mathcal{Z}))$
- Binary trees: $\mathcal{B} = \mathcal{Z} + \mathcal{B} \times \mathcal{B}$
- General (Cayley) trees: $\mathcal{T} = \mathcal{Z} \times \text{SET}(\mathcal{T})$
- Functional graphs: $\mathcal{F} = \text{SET}(\mathcal{K}), \mathcal{K} = \text{CYC}(\mathcal{T}), \mathcal{T} = \mathcal{Z} \times \text{SET}(\mathcal{T})$



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- Are these necessary conditions?
 - ▷ Yes, if we do not authorize structures of size 0 and zero-coordinates.
- Can we give a characterization that allows structures of size 0?

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  F01 = FA01,
  Sorig = Prod(L, LA),
  S0 = Union(LA0, Prod( Prod(Union(Epsilon, Epsilon), S0), LA0)),
  S00 = Union(LA00, Prod( Prod(Union(Epsilon, Epsilon), S0), LA00)),
  S01 = Union(SA01, Prod( Prod(Union(Epsilon, Epsilon), S0), LA01)),
  FA = Epsilon,
  FA01 = Prod(t, Union(F, Prod(L, LA0))),
  FA0011 = Prod(t, Union(F01, Prod(L, LA011))),
  FA0101 = Prod(t, Union(F01, Prod(L, LA010))),
  FA0110 = Prod(t, Prod(L, LA001)),
  LA = Prod(Union(Epsilon, Epsilon), LA0),
  LA0 = Prod(t, Union(Union(Sorig, F), Prod(L, Union(Union(Union(LA0, GA0), GA010), GA011), GA001)))),
  LA00 = Prod(t, Union(S0, Prod(L, Union(LA00, GA011)))),
  LA01 = Prod(t, Union(Union(S0, F), Prod(L, Union(Union(Union(LA01, GA001), GA010), GA0))),
  LA000 = Prod(t, Union(S00, Prod(L, LA000))),
  LA001 = Prod(t, Union(S00, Prod(L, Union(LA001, GA001)))),
  LA010 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA010, GA010)))),
  LA011 = Prod(t, Union(Union(S01, F01), Prod(L, Union(LA011, GA011))),
  SA = Prod(Union(Epsilon, Epsilon), LA0),
  SA01 = Prod(t, Union(S0, Prod(L, Union(Union(SA01, GA001), GA010))) ),
  GA0 = Prod(t, Union(Union(Union(Sorig, Prod(L, GA0)), Prod(L, GA010)), Prod(L, GA011)), Prod(L, GA001))),
  GA001 = Prod(t, Union(S00, Prod(L, GA001))),
  GA010 = Prod(t, Union(S01, Prod(L, GA010))),
  GA011 = Prod(t, Union(S01, Prod(L, GA011))),
  t=Atom):
```

- **Sufficient** conditions for a system to define combinatorial species?
 - ▷ $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\partial\mathcal{H}/\partial\mathcal{Y}(\mathbf{0}, \mathbf{0})$ is nilpotent [Joyal, 1981].
- Are these **necessary** conditions?
 - ▷ Yes, if we do not authorize structures of size 0 and zero-coordinates.
- Can we give a characterization that allows structures of size 0?
 - ▷ Yes, we have two different characterizations.

Well-founded systems

Definition (Well founded)

The combinatorial system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ is *well founded* when

$$\mathcal{Y}^{[0]} = \mathbf{0} \quad \text{and} \quad \mathcal{Y}^{[n+1]} = \mathcal{H}(\mathcal{Z}, \mathcal{Y}^{[n]}), \quad n \geq 0$$

is well defined for all n and defines a sequence $(\mathcal{Y}^{[n]})_{n \geq 0}$ that is convergent. Its limit \mathcal{S} is *the combinatorial solution* of the system.

- *well defined*: for each n and each size k , there are finitely many combinatorial structures of size k in $\mathcal{H}(\mathcal{Z}, \mathcal{Y}^{[n]})$;
- *convergent*: there is a limit class \mathcal{S} such that for all $k \geq 0$, $\mathcal{Y}^{[n]}$ is isomorphic to \mathcal{S} for all structures of size up to k , for large enough n .

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Theorem (Characterization by Leading Terms)

The system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ is well founded if and only if the **leading terms** (smallest structures) of $\mathcal{Y}^{[m]}$ and $\mathcal{Y}^{[m+1]}$ are equal.

Computing the leading terms

Algorithm WellFoundedAndLeadingTerm:

Input: $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ a specification in **normal form** with m equations

Output: Vector of pairs (c, v) corresponding to the vector of leading terms $c\mathcal{Z}^v$ of \mathcal{Y} ;
FAIL if and only if the system is not well founded

```
1  $w_{1:m} := ((0, \infty), \dots, (0, \infty))$ 
2 repeat  $m+1$  times
3    $v := w$ 
4   for  $i = 1$  to  $m$  do
5     if  $\mathcal{H}_i = 1$  then  $w_i := (1, 0)$ 
6     if  $\mathcal{H}_i = \mathcal{Z}$  then  $w_i := (1, 1)$ 
7     if  $\mathcal{H}_i = \text{SET}(\mathcal{Y}_j)$  or  $\text{SEQ}(\mathcal{Y}_j)$  then  $w_i := (1, 0)$  if  $v_{j,2} \neq 0$ , otherwise FAIL
8     if  $\mathcal{H}_i = \text{CYC}(\mathcal{Y}_j)$  then  $w_i := v_j$  if  $v_{j,2} \neq 0$ , otherwise FAIL
9     if  $\mathcal{H}_i = \mathcal{Y}_{j_1} \times \dots \times \mathcal{Y}_{j_k}$  then  $w_i := (v_{j_1,1} \dots v_{j_k,1}, v_{j_1,2} + \dots + v_{j_k,2})$ 
10    if  $\mathcal{H}_i = \mathcal{Y}_{j_1} + \dots + \mathcal{Y}_{j_k}$  then
11       $w_{i,2} := \min(v_{j_1,2}, \dots, v_{j_k,2}), w_{i,1} := \sum_{\ell \in [1,k], v_{j_\ell,2} = w_{i,2}} v_{j_\ell,1}$ 
12    if  $v = w$  then return  $v$  ◁ the system is well founded
13 if  $v \neq w$  then FAIL
14 return  $v$ 
```

Improvement of Paul Zimmermann's algorithm (1991)

A system not too big, but not that small

Specification for unordered forests of trees with red, green and blue nodes and its normal form.

$$\left\{ \begin{array}{l} \mathcal{F} = \text{SET}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{R} \times \text{SEQ}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \text{SEQ}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \text{SEQ}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{Z}^3 + \mathcal{Z} \times \text{SEQ}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \text{SEQ}(\mathcal{R}), \\ \mathcal{G} = \mathcal{Z} + \mathcal{G} \times \mathcal{G}. \end{array} \right. \quad \left\{ \begin{array}{l} \mathcal{F} = \text{SET}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{N}_1, \quad \mathcal{N}_1 = \mathcal{R} \times \mathcal{N}_2, \quad \mathcal{N}_2 = \text{SEQ}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \mathcal{N}_3, \quad \mathcal{N}_3 = \text{SEQ}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \mathcal{N}_4, \quad \mathcal{N}_4 = \text{SEQ}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{N}_5 + \mathcal{N}_6, \quad \mathcal{N}_5 = \mathcal{Z}^3, \quad \mathcal{N}_6 = \mathcal{Z} \times \mathcal{N}_7, \quad \mathcal{N}_7 = \text{SEQ}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \mathcal{N}_8, \quad \mathcal{N}_8 = \text{SEQ}(\mathcal{R}), \\ \mathcal{G} = \mathcal{Z} + \mathcal{N}_9, \quad \mathcal{N}_9 = \mathcal{G} \times \mathcal{G} \end{array} \right.$$

The algorithm produces the following vectors, the system is well founded:

$$\begin{aligned} & [1, \mathcal{Z}, 0, 1, 0, 1, 0, 1, 0, \mathcal{Z}^3, 0, 1, 0, 1, \mathcal{Z}, 0] \\ & [1, \mathcal{Z}, 0, 1, 0, 1, \mathcal{Z}, 1, \mathcal{Z}^3, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\ & [1, \mathcal{Z}, \mathcal{Z}^3, 1, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\ & [1, \mathcal{Z}, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\ & [1, 2\mathcal{Z}, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \\ & [1, 2\mathcal{Z}, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, \mathcal{Z}, \mathcal{Z}^2] \end{aligned}$$

A different system not too big, but not that small

Specification for unordered forests of trees with red, green and blue nodes and its normal form.

$$\left\{ \begin{array}{l} \mathcal{F} = \text{SET}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{R} \times \text{SEQ}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \text{SEQ}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \text{SEQ}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{Z}^3 + \mathcal{Z} \times \text{SEQ}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \text{SEQ}(\mathcal{R}), \\ \mathcal{G} = 1 + \mathcal{Z} \times \mathcal{G}^2. \end{array} \right. \quad \left\{ \begin{array}{l} \mathcal{F} = \text{SET}(\mathcal{T}_r), \\ \mathcal{T}_r = \mathcal{Z} + \mathcal{N}_1, \quad \mathcal{N}_1 = \mathcal{R} \times \mathcal{N}_2, \quad \mathcal{N}_2 = \text{SEQ}(\mathcal{T}_b), \\ \mathcal{T}_b = \mathcal{B} \times \mathcal{N}_3, \quad \mathcal{N}_3 = \text{SEQ}(\mathcal{T}_g), \\ \mathcal{T}_g = \mathcal{G} \times \mathcal{N}_4, \quad \mathcal{N}_4 = \text{SEQ}(\mathcal{T}_r), \\ \mathcal{R} = \mathcal{N}_5 + \mathcal{N}_6, \quad \mathcal{N}_5 = \mathcal{Z}^3, \quad \mathcal{N}_6 = \mathcal{Z} \times \mathcal{N}_7, \quad \mathcal{N}_7 = \text{SEQ}(\mathcal{B}), \\ \mathcal{B} = \mathcal{Z} \times \mathcal{N}_8, \quad \mathcal{N}_8 = \text{SEQ}(\mathcal{R}), \\ \mathcal{G} = 1 + \mathcal{N}_9, \quad \mathcal{N}_9 = \mathcal{Z} \times \mathcal{G} \times \mathcal{G} \end{array} \right.$$

At some point, the leading term of \mathcal{T}_g is 1. This breaks the condition of being well defined for $\mathcal{N}_3 = \text{SEQ}(\mathcal{T}_g)$ in the next iteration. The system is not well founded anymore.

$$\begin{aligned} & [1, \mathcal{Z}, 0, 1, 0, 1, 0, 1, 0, \mathcal{Z}^3, 0, 1, 0, 1, 1, 0] \\ & [1, \mathcal{Z}, 0, 1, 0, 1, \mathbf{1}, 1, \mathcal{Z}^3, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, 1, \mathcal{Z}] \\ & \quad \Delta \\ & \quad \mathcal{T}_g \end{aligned}$$

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The previous condition is not broken anymore... But the last 2 vectors after $m + 1 = 18$ iterations are not equal. This system is (still) not well founded:

$$[1, \mathcal{Z}, 0, 1, 0, 1, 0, 0, 1, 0, \mathcal{Z}^3, 0, 1, 0, 1, 1, 0]$$

...

$$[1, 2\mathcal{Z}, \mathcal{Z}, 1, 8\mathcal{Z}, 8, 8, 1, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, 1, \mathcal{Z}]$$

$$[1, 2\mathcal{Z}, \mathcal{Z}, 1, 8\mathcal{Z}, 9, 8, 1, 1, \mathcal{Z}, \mathcal{Z}^3, \mathcal{Z}, 1, \mathcal{Z}, 1, 1, \mathcal{Z}]$$

Δ

\mathcal{N}_3

Generating functions

Exponential generating functions for labeled structures

The exponential generating function of a species $\mathcal{Y}_{1:m}$ is the power series

$$Y(z) = \sum_{n \geq 0} Y_n \frac{z^n}{n!}$$

where Y_n is the vector of numbers of \mathcal{Y}_k -structures of size n , for $k = 1..m$.

species	constr.	exponential g.f.	derivative $\partial H/\partial z$
Union	$\mathcal{F} + \mathcal{G}$	$F(z) + G(z)$	$F'(z) + G'(z)$
Product	$\mathcal{F} \times \mathcal{G}$	$F(z) \times G(z)$	$F'(z)G(z) + F(z)G'(z)$
Sequence	$\text{SEQ}(\mathcal{G})$	$(1 - G(z))^{-1}$	$(1 - G(z))^{-2}$
Cycle	$\text{CYC}(\mathcal{G})$	$\ln \frac{1}{1 - G(z)}$	$(1 - G(z))^{-1}$
Set	$\text{SET}(\mathcal{G})$	$e^{G(z)}$	$e^{G(z)}$

Any specification translates into a system of equations on generating series.

Jacobian matrix, dependency graph, and irreducible systems

$$\begin{cases} F(z) = e^{T_r(z)} \\ T_r(z) = z + \frac{R(z)}{1-T_b(z)} \\ T_b(z) = \frac{B(z)}{1-T_g(z)} \\ T_g(z) = \frac{G(z)}{1-T_r(z)} \\ R(z) = z^3 + \frac{z}{1-B(z)} \\ B(z) = \frac{z}{1-R(z)} \\ G(z) = z + G(z)^2 \end{cases}$$

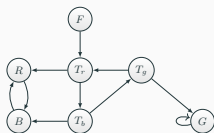
Jacobian matrix, dependency graph, and irreducible systems

$$\left\{ \begin{array}{l} F(z) = e^{T_r(z)} \\ T_r(z) = z + \frac{R(z)}{1-T_b(z)} \\ T_b(z) = \frac{B(z)}{1-T_g(z)} \\ T_g(z) = \frac{G(z)}{1-T_r(z)} \\ R(z) = z^3 + \frac{z}{1-B(z)} \\ B(z) = \frac{z}{1-R(z)} \\ G(z) = z + G(z)^2 \end{array} \right. \quad \begin{pmatrix} 0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\ 0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\ 0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2G(z) \end{pmatrix}$$

Jacobian matrix, dependency graph, and irreducible systems

$$\begin{cases} F(z) = e^{T_r(z)} \\ T_r(z) = z + \frac{R(z)}{1-T_b(z)} \\ T_b(z) = \frac{B(z)}{1-T_g(z)} \\ T_g(z) = \frac{G(z)}{1-T_r(z)} \\ R(z) = z^3 + \frac{z}{1-B(z)} \\ B(z) = \frac{z}{1-R(z)} \\ G(z) = z + G(z)^2 \end{cases} \quad \begin{pmatrix} 0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\ 0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\ 0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2G(z) \end{pmatrix}$$

The Jacobian matrix $\partial H / \partial Y$ encodes the dependency graph of the system.



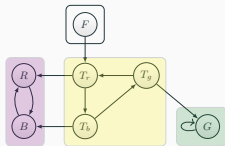
Dependency graph

Jacobian matrix, dependency graph, and irreducible systems

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$$\begin{pmatrix} 0 & e^{T_r(z)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R(z)}{(1-T_b(z))^2} & 0 & \frac{1}{1-T_b(z)} & 0 & 0 \\ 0 & 0 & 0 & \frac{B(z)}{(1-T_g(z))^2} & 0 & \frac{1}{1-T_g(z)} & 0 \\ 0 & \frac{G(z)}{(1-T_r(z))^2} & 0 & 0 & 0 & 0 & \frac{1}{1-T_r(z)} \\ 0 & 0 & 0 & 0 & 0 & \frac{z}{(1-B(z))^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{z}{(1-R(z))^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2G(z) \end{pmatrix}$$

The Jacobian matrix $\partial H / \partial Y$ encodes the dependency graph of the system.



Strongly connected components

Definition (Irreducible)

A combinatorial system is *irreducible* when its dependency graph is strongly connected.

The analysis of general combinatorial systems often reduces to the special case of irreducible systems.

Well-founded systems of generating functions

Definition

Let $\mathbf{H}_{1:m} : \mathbb{C}^{m+1} \rightarrow \mathbb{C}^m$ be analytic in a neighborhood of $\mathbf{0}$.

The system $\mathbf{Y} = \mathbf{H}(z, \mathbf{Y})$ is called *well founded* when

1. Each coordinate of \mathbf{H} has nonnegative real Taylor coefficients at $\mathbf{0}$.
2. For $k = 0, \dots, m$, the iteration

$$\mathbf{U}^{[0]} := \mathbf{0}, \quad \mathbf{U}^{[k+1]} := \mathbf{H}(0, \mathbf{U}^{[k]})$$

that computes g.f. for the structures of size 0, is well defined.

3. The Jacobian matrix $\partial \mathbf{H} / \partial \mathbf{Y}$ is nilpotent at $(0, \mathbf{U}^{[m]})$.

This is by analogy with our extension of [Joyal's implicit species theorem](#) for combinatorial systems without the constraint $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Generating functions solutions of well-founded systems

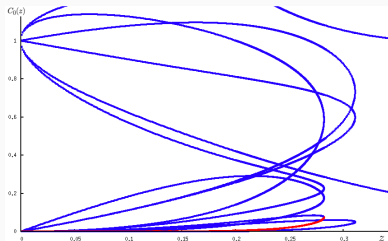
$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$

$$C_1(z) = z + \frac{z}{1 - C_1(z)^2 C_3(z)^2}$$

$$C_2(z) = z + \frac{z^2}{(1 - zC_2(z)^2 / (1 - z))(1 - C_2(z))}$$

$$C_3(z) = z + \frac{z(3z + z^2 + z^2 C_1(z)C_3(z))}{1 - C_1^2(z)}$$

A system of generating function equations may have many solutions...



Generating functions solutions of well-founded systems

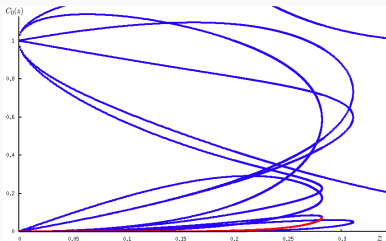
$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$

$$C_1(z) = z + \frac{z}{1 - C_1(z)^2 C_3(z)^2}$$

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A system of generating function equations may have many solutions...



The only solution of interest is the one that corresponds to the value of the generating function of \mathcal{C} .

Generating functions solutions of well-founded systems

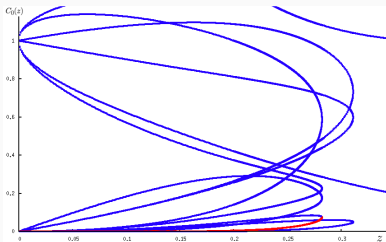
$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z)+C_2(z))$$

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A system of generating function equations may have many solutions...



The only solution of interest is the one that corresponds to the value of the generating function of \mathcal{C} .

Definition (generating function solution)

If $Y = H(z, Y)$ is *well-founded*, then it admits a **unique** solution $Y(z)$ analytic in a neighborhood of 0 with $Y(0) = U^{[m]}$. This solution has nonnegative Taylor coefficients at 0.

$Y(z)$ is called the *generating function solution* of the system.

Recall that $U^{[m]}$ is the generating function for the structures of size 0.

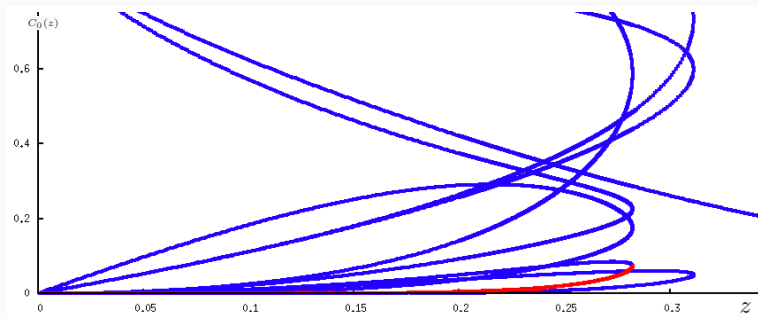
Second part: Analytic Functions

The centerpiece: computing the radius

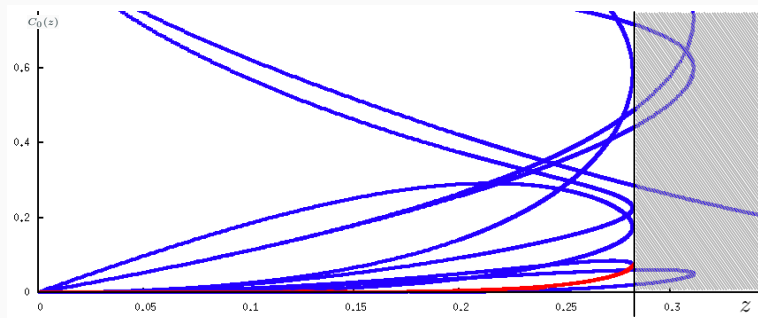
Computation of the radius of convergence

First Principle of Coefficient Asymptotics: The location of a function's singularities dictates the **exponential growth** of its coefficients [Flajolet and Sedgewick, 2009].

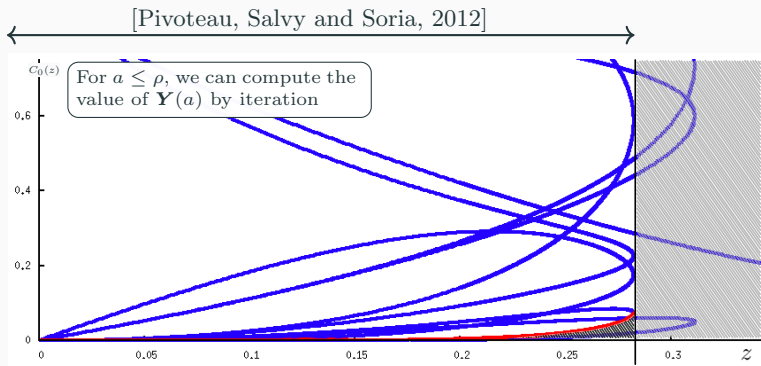
Computation of the radius of convergence



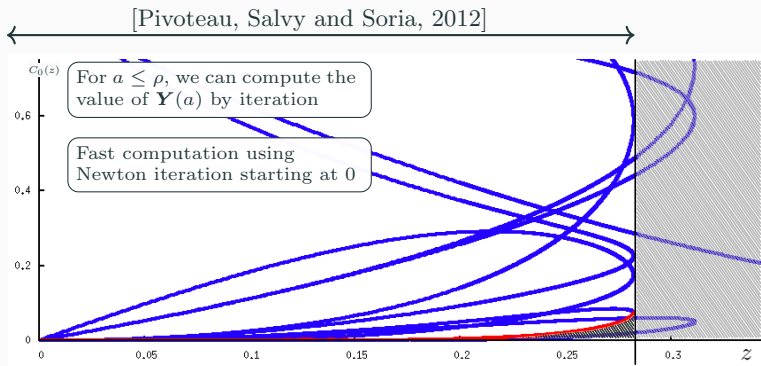
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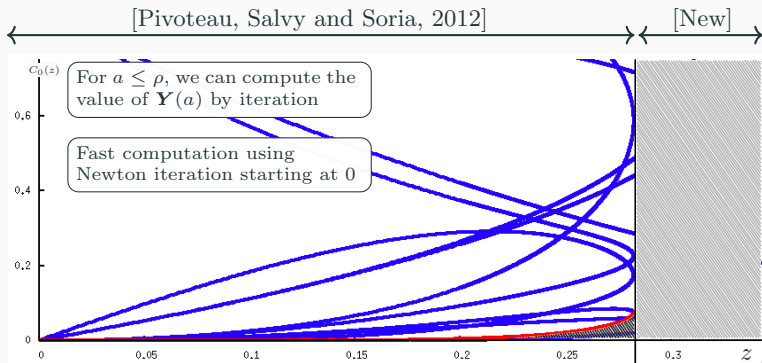
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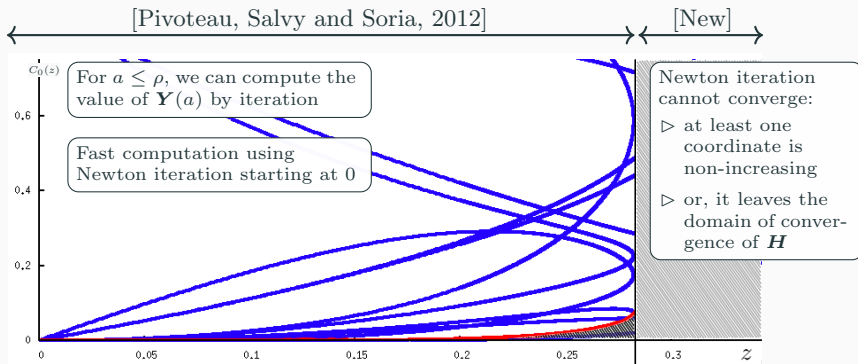
Theorem (Value of the generating function solution)

Let $\mathbf{Y} = \mathbf{H}(z, \mathbf{Y})$ be a well-founded system and let ρ be the radius of convergence of its g.f. solution $\mathbf{Y}(z)$. Let $(a, \mathbf{B}) \in \mathbb{R}_{\geq 0}^{m+1}$ belong to the domain of convergence of \mathbf{H} or its boundary and satisfy $\mathbf{B} = \mathbf{H}(a, \mathbf{B})$.

Then $\rho > 0$, $a \leq \rho$, $\mathbf{B} \geq \mathbf{Y}(a)$ and, moreover, $\Lambda_{\mathbf{H}}(a, \mathbf{B}) \leq 1 \Leftrightarrow \mathbf{B} = \mathbf{Y}(a)$.

$\Lambda_{\mathbf{H}}(x, \mathbf{y})$ is the spectral radius of the Jacobian matrix $\partial \mathbf{H} / \partial \mathbf{Y}$.

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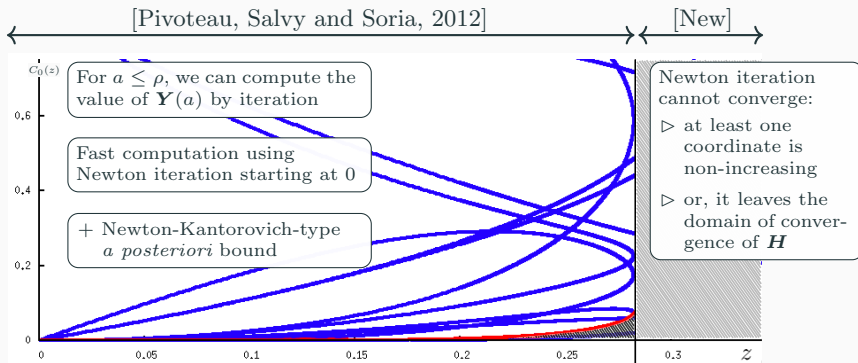


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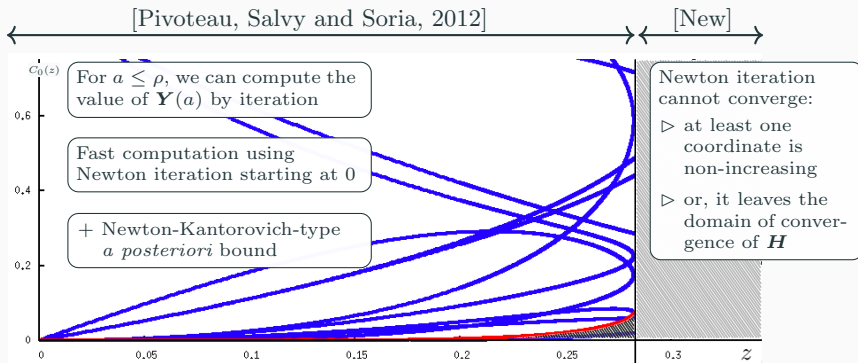


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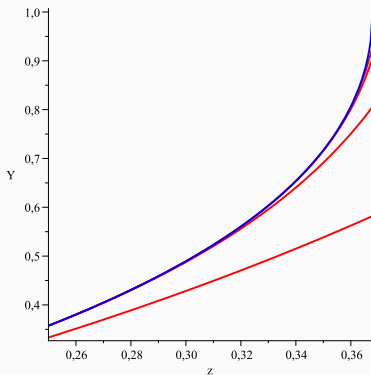
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▷ We can compute the radius of convergence **by dichotomy**...

Another way to compute the radius

... by dichotomy, but Newton iteration is slow when a is close to ρ .



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But it may have several solutions... For instance, this characteristic system

$$\{y_1 - z(y_1^2 + y_2 + 1), y_2 - z(y_2^2 + y_1 + 1), 4y_1y_2z^2 - 2y_1z - 2y_2z - z^2 + 1\}$$

has 2 solutions for (z, y_1, y_2) , $(1 \ 3 \ 1 \ 1)$ and $((2\sqrt{2} - 1) \ 7 \ \sqrt{2} + 1 \ \sqrt{2} + 1)$.

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▷ We can filter them with the criterion $\Lambda_{\mathbf{H}}(a, \mathbf{B}) = 1$.

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Decompose the system into **irreducible components**, and

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- use Newton iteration started at $(a, \mathbf{Y}(a))$ on the characteristic system;
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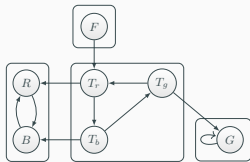
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On the previous example, starting at $a = 0.05375$ and the values of $y_1(a)$ and $y_2(a)$, Newton's iteration on the characteristic system gives (for (y_1, z)):

$(0.0569876, 0.05375), (1.98856, 1.76650), (-2.49792, 5.32162), (-1.25430, 5.31255), \dots$
 $(1.00003, 0.317617), (0.998510, 0.333309), (0.999996, 0.333333), (\mathbf{1.00000}, \mathbf{0.333333})$.

Computation of the radius for colored trees

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 - $G(z) = \frac{1 - \sqrt{1 - 4z}}{2} \rightarrow \rho_G = 1/4.$



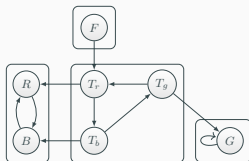
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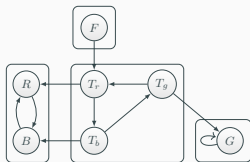
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- the radius is either the minimum of ρ_G and ρ_{BR} , or smaller (where Λ should be 1).
 - Newton's iteration does not converge at 0.2462661 for S_T : $\rho < \rho_{BR}$
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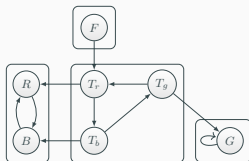
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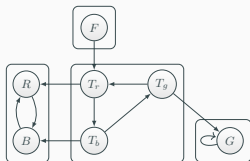
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 - $\Lambda_{S_T}(0.2462661) \approx 0.127 < 1$.
 - This implies that $\rho = \rho_{BR} \in [0.2462661, 0.2462662]$, from the component S_{BR} .
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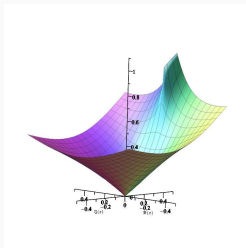
The last pieces fall into place

First Principle of Coefficient Asymptotics:

The location of a function's singularities dictates the exponential growth of its coefficients.

Second Principle of Coefficient Asymptotics:

The nature of a function's singularities determines the associated [subexponential factor](#).

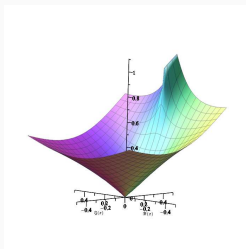


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Next steps:

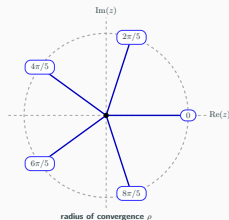
- locate all the **dominant singularities**;
- computation of the **local expansions** of the generating functions in the neighborhood of these singularities;
- **asymptotic behavior of the coefficients** using transfer theorems.

Dominant singularities

The radius of convergence ρ is a dominant singularity.

There might be others on the circle $|z| = \rho$.

Their arguments are **rational multiples of π** , induced by periodicity properties that can be computed from the combinatorial equations.

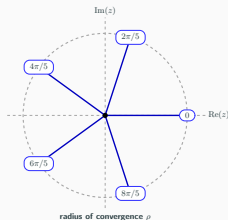


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We extend scattered known results to all the well-founded specifications:

- An algorithm to **compute the periods** of the generating functions.
 - ▷ Valuations + fixed-point iteration
- **Decompose the system** into strongly connected components and proceed bottom-up, restricting to the components having ρ as radius.
- Treating separately:
 - nonrecursive components
 - irreducible **linear/nonlinear** components

Dominant singularities of irreducible systems: examples

Nonlinear case:

$$\mathcal{A} = \mathcal{Z}^2 + \mathcal{Z}^2 \times \mathcal{B}^2, \quad \mathcal{B} = \mathcal{Z}^3 + \mathcal{Z}^3 \times \mathcal{A}^2.$$



The valuations are $v_A = 2$ and $v_B = 3$ and the periods are solutions to the system $q_A = \gcd(6, q_B)$, $q_B = \gcd(4, q_A)$.

Our algorithm gives $(0, 0) \mapsto (6, 4) \mapsto (2, 2) \mapsto (2, 2)$, which places dominant singularities at arguments at $e^{2ik\pi/2}$ for $k = 0$ and $k = 1$.

Linear case:

$$\mathcal{A} = \mathcal{Z}^2 + \mathcal{Z}^2 \times \mathcal{B}, \quad \mathcal{B} = \mathcal{Z}^3 + \mathcal{Z}^3 \times \mathcal{A}.$$

The equations for the periods are $q_A = \gcd(3, q_B)$ and $q_B = \gcd(2, q_A)$, which gives $q_A = q_B = 1 \dots$ but the generating functions solutions are

$$A(z) = z^2 \frac{1+z^3}{1-z^5}, \quad B(z) = z^3 \frac{1+z^2}{1-z^5}, \dots$$



... which indicates dominant singularities at $e^{2ik\pi/5}$ for $k = 0, \dots, 4$.

Dominant singularities of irreducible systems: algorithm

Algorithm Dominant singularities of the generating function:

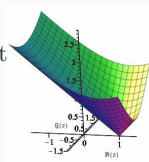
Input: $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{U}, \mathcal{Y})$ an irreducible constructive component with $\partial\mathcal{H}/\partial\mathcal{U} \neq 0$;
 R radius of convergence of its generating function solution \mathcal{Y} ;
 $S_{\mathcal{U}} := \mathcal{U} = \mathcal{G}(\mathcal{Z}, \mathcal{U})$
 $\Delta_{\mathcal{U}} := \bigcup_{\mathcal{U} \in \mathcal{U}} \Delta_{\mathcal{U}}(R)$ with $\Delta_{\mathcal{U}}(R) \supseteq \{\alpha \mid Re^{i\alpha} \text{ singularity of } U\}$

Output: $S \supseteq \{\alpha \mid Re^{i\alpha} \text{ singularity of } \mathcal{Y}\}$

```
1 if  $\partial^2\mathcal{H}/\partial\mathcal{Y}^2 \neq 0$  then ◁ nonlinear case
2    $q := \text{Periods}(\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{U}, \mathcal{Y}), S_{\mathcal{U}})$ 
3   return  $\{2k\pi/q_1 \mid k = 0, \dots, q_1 - 1\}$  ◁ Theorems
4 else ◁ linear case
5   if  $\Lambda_H(R) < 1$  then ◁ inequality
6     return  $\Delta_{\mathcal{U}}$ 
7   else
8     Define  $\mathcal{A}(\mathcal{Z}, \mathcal{U}), \mathcal{B}(\mathcal{Z}, \mathcal{U})$  by  $\mathcal{H} = \mathcal{A} + \mathcal{B}\mathcal{Y}$  ◁  $\mathcal{Y} = (\text{Id} - \mathcal{B})^{-1}\mathcal{A}$ 
9      $q := \text{gcd}(\text{Periods}(\mathcal{Y} = \mathcal{B}(\mathcal{Z}, \mathcal{U}), S_{\mathcal{U}}))$ 
10    if  $q = 0$  then
11       $Q := \text{Periods}(\mathcal{Y} = \text{Id} + \mathcal{B}(\mathcal{Z}, \mathcal{U})\mathcal{Y}, S_{\mathcal{U}})$ 
12       $q := \text{gcd}(\text{diagonal}(Q))$ 
13    return  $\Delta_{\mathcal{U}} \cup \{2k\pi/q \mid k = 0, \dots, q - 1\}$  ◁ Theorems
```

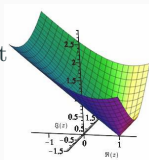
We **classify the singular behaviors** for all constructive systems and prove a **gap property**: the generating function...

- ... is either of **superpolynomial growth**: it grows at least as fast as $\exp(c \ln^2(1 - z/\rho))$ for some $c > 0$.
- or it has an **algebraic-logarithmic** behavior.
- No other behavior is possible.



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In the algebraic-logarithmic case:

- Either it behaves like:

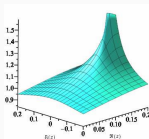
$$c_0 + c_1(1 - z/\rho)^{1/r} + c_2(1 - z/\rho)^{2/r} + \dots$$

where r is a power of 2 [Banderier and Drmota, 2015].

- Otherwise, it has an infinite limit and is equivalent to

$$C(1 - z/\rho)^\alpha \ln^k\left(\frac{1}{1 - z/\rho}\right), \quad C > 0, \alpha \leq 0, k \in \mathbb{N}, (\alpha, k) \neq (0, 0).$$

- More complex exponents may occur at other singularities.



Example: the generating function of $\mathcal{Y} = \text{SET}(\mathcal{Z} \times \text{CYC}(\mathcal{Z}^4))$ is

$$Y(z) = \exp\left(z \ln\left(\frac{1}{1-z^4}\right)\right)$$

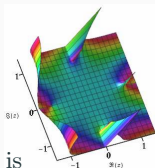
and has 4 dominant singularities at $\{\pm 1, \pm i\}$. Its local behavior is

$$f(z) = \frac{1/4}{1-z} - \frac{1}{4} \ln\left(\frac{1}{1-z}\right) + \frac{3}{8} + \frac{\ln 2}{2} + O\left((1-z) \ln^2\left(\frac{1}{1-z}\right)\right), \quad z \rightarrow 1,$$

$$f(z) = 4(1+z) + O\left((1+z)^2 \ln\left(\frac{1}{1+z}\right)\right), \quad z \rightarrow -1,$$

$$f(z) = 4^{-i}(1-z/i)^{-i} + O\left((1-z/i)^{-i+1} \ln\left(\frac{1}{1-z/i}\right)\right), \quad z \rightarrow i$$

and the conjugate of the last one as $z \rightarrow -i$.



Example: the generating function of $\mathcal{Y} = \text{SET}(\mathcal{Z} \times \text{CYC}(\mathcal{Z}^4))$ is

$$Y(z) = \exp\left(z \ln\left(\frac{1}{1-z^4}\right)\right)$$

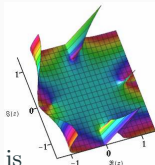
and has 4 dominant singularities at $\{\pm 1, \pm i\}$. Its local behavior is

$$f(z) = \frac{1/4}{1-z} - \frac{1}{4} \ln\left(\frac{1}{1-z}\right) + \frac{3}{8} + \frac{\ln 2}{2} + O\left((1-z) \ln^2\left(\frac{1}{1-z}\right)\right), \quad z \rightarrow 1,$$

$$f(z) = 4(1+z) + O\left((1+z)^2 \ln\left(\frac{1}{1+z}\right)\right), \quad z \rightarrow -1,$$

$$f(z) = 4^{-i}(1-z/i)^{-i} + O\left((1-z/i)^{-i+1} \ln\left(\frac{1}{1-z/i}\right)\right), \quad z \rightarrow i$$

and the conjugate of the last one as $z \rightarrow -i$.



Theorem (Singular behavior)

Consider a constructive system and $\sigma \in \mathbb{C}$ that lies inside or on the boundary of the domain of convergence of its generating function solution.

Our **algorithm** computes truncations of the expansions of all its coordinates that are analytic or have an algebraic-logarithmic singularity at σ . It returns EXP for the other coordinates.

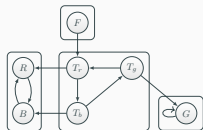
Asymptotic behavior

The expansion of $F(z)$ at its dominant singularity ρ is of the form

$$c_0 + c_1 u^{1/2} + c_2 u + c_3 u^{3/2} + \dots,$$

with $u = 1 - z/\rho$. Numerical values can be refined to arbitrary precision:

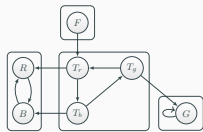
$$F = 1.801525 - 0.892560 u^{1/2} + 0.044708 u + 0.044641 u^{3/2} + \dots$$



Asymptotic behavior

The expansion of $F(z)$ at its dominant singularity ρ is of the form

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$$F = 1.801525 - 0.892560 u^{1/2} + 0.044708 u + 0.044641 u^{3/2} + \dots$$

By singularity analysis, the **asymptotic behavior of the coefficients** of $F(z)$ is deduced directly from this expansion. The n th coefficient behaves like:

$$\frac{\rho^{-n}}{n!} \left(c_1 \frac{\Gamma(n-1/2)}{\Gamma(-1/2)} + c_3 \frac{\Gamma(n-3/2)}{\Gamma(-3/2)} + c_5 \frac{\Gamma(n-5/2)}{\Gamma(-5/2)} + O(\Gamma(n-7/2)) \right), \quad n \rightarrow \infty.$$

With the numerical values and $\rho \in [0.1703916, 0.1703917]$:

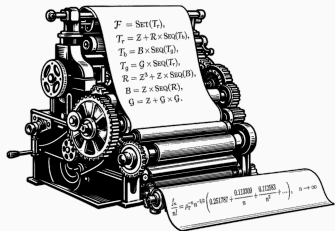
$$\frac{f_n}{n!} = \rho^{-n} n^{-3/2} \left(0.251787 + \frac{0.113309}{n} + \frac{0.112583}{n^2} + \dots \right) \quad n \rightarrow \infty$$

Theorem (Asymptotic behavior)

Let \mathbf{Y} be a constructible generating function. An **asymptotic expansion** can be computed for all Y_i that have an algebraic-logarithmic dominant singularity.

Conclusion

We give an almost **complete algorithmic chain** going from combinatorial systems to **asymptotic expansions** for all the exponential generating functions produced by the symbolic method of Flajolet and Sedgewick.

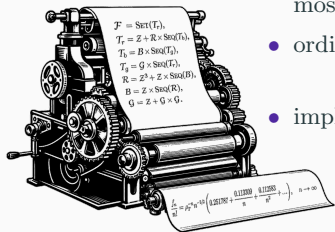


Conclusion

We give an almost **complete algorithmic chain** going from combinatorial systems to **asymptotic expansions** for all the exponential generating functions produced by the symbolic method of Flajolet and Sedgewick.

There is still work to be done:

- superpolynomial growth;
- cardinality constraints for SEQ, SET, CYC; mostly technical;
- ordinary generating functions: for some simple cases;
- implementation: Maple worksheets, but no library.



The End

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Decidability issues

Our algorithms need to decide equality between real values. This is a difficult question in general. We define an oracle tailored for our needs:

Constant Oracle

Given a constructive system, the *Constant Oracle* can

- decide whether $\mathbf{Y} = \mathbf{H}(z, \mathbf{Y})$ has a solution, for a fixed $z \geq 0$;
- if so, decide whether it has a solution with $\Lambda(z, \mathbf{Y}) = 1$;
- test equality or inequality between solutions of such systems.

Proposition

If the radius of convergence of a constructible generating function is finite, then it can be computed with the help of the Constant Oracle.